p-adic Hodge theory

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(ii) There is a natural isomorphism

$$H^i(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^i_{\mathrm{dR}}(X) \ .$$

Classical Hodge theory

(iii) If X is Kähler, then the Hodge-de Rham spectral sequence

$$E_1^{ij} = H^j(X, \Omega_X^i) \Rightarrow H_{\mathrm{dR}}^{i+j}(X)$$

degenerates at E_1.

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be the unipotent subgroup of GL_3. Then

$$X := N(\mathbb{C})/N(\mathbb{Z}[i]) \ .$$

The Hodge-de Rham spectral does not degenerate at E_1; in particular X is non-Kähler.

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Theorem (Tate, 1967)

Let $A/\mathcal{O}_K$ be an abelian variety. Then there is a natural $\mathrm{Gal}(\bar{K}/K)$-equivariant isomorphism

$$H^1_{\text{ét}}(A_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C \cong H^1(A, \mathcal{O}_A) \otimes_K C \oplus H^0(A, \Omega^1_A)(-1) \otimes_K C \ .$$

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Such "Hodge-like" decompositions are now called Hodge-Tate decompositions.

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Rigid-analytic case remained open. Not even finiteness of $H^i_{\text{ét}}(X_C, \mathbb{Z}_p)$ was known!

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Thus, there exist "non-Kähler" rigid-analytic varieties. However, there is no p-adic analogue of the Iwasawa manifold. Namely, C has no cocompact discrete subgroups like $\mathbb{Z}[i] \subset \mathbb{C}$.

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(i) *For all $i \geq 0$, $H^i_{\text{ét}}(X_C, \mathbb{Z}_p)$ is a finitely generated $\mathbb{Z}_p$-module, which is zero for $i > 2 \dim X$.*

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(iii) *There is a natural* $\mathrm{Gal}(\bar{K}/K)$*-equivariant de Rham comparison isomorphism*

$$H^i_{\text{ét}}(X_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}} \cong H^i_{\mathrm{dR}}(X) \otimes_K B_{\mathrm{dR}}$$

preserving filtrations; in particular, $H^i_{\text{ét}}(X_C, \mathbb{Q}_p)$ *is de Rham in the sense of Fontaine.*

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(iv) *The Hodge-de Rham spectral sequence*

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On finiteness for $\mathbb{Z}_p$-cohomology

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Need global argument, using finiteness of coherent cohomology to control Artin-Schreier type covers.

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where $\overline{x} \in X$ is a geometric base point, and $\pi_1(X, \overline{x})$ is the profinite étale fundamental group.

$K(\pi, 1)$'s, equal characteristic case

Also true for X over equal characteristic field $\widehat{k((t))}$.

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$$H^i_{\text{ét}}(X, \mathbb{F}_p) = 0$$

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Moreover, we get a long exact sequence

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where $R = H^0(X, \mathcal{O}_X)$.

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where $R = H^0(X, \mathcal{O}_X)$. As exactness of Artin-Schreier sequence needs only finite étale covers, get same result for

$$H^i_{\text{cont}}(\pi_1(X, \overline{x}), \mathbb{L}_{\overline{x}}) \ .$$

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Recall Fontaine's field

$$C^\flat = \mathcal{R}(C) = \mathrm{Frac}\,(\varprojlim_{\Phi} \mathcal{O}_C/p) \cong \widehat{\overline{k((\varpi))}}\ .$$

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  (Iterate adjoining p-power roots of units.)

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- Its tilt $\tilde{X}^\flat$ is affinoid and lives in equal characteristic, hence is a $K(\pi, 1)$ for p-torsion coefficients.

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  (Iterate adjoining p-power roots of units.)
- Its tilt $\tilde{X}^\flat$ is affinoid and lives in equal characteristic, hence is a $K(\pi, 1)$ for p-torsion coefficients.
- As $\tilde{X}_{\text{ét}} \cong \tilde{X}^\flat_{\text{ét}}$, also $\tilde{X}$ is a $K(\pi, 1)$ for p-torsion coefficients.

$K(\pi, 1)$'s, mixed characteristic case

Want to show: Any connected affinoid rigid-analytic variety X over C is a $K(\pi, 1)$.

Proof.

- Find affinoid perfectoid space $\tilde{X}$ over C, $\tilde{X} \to X$ an inverse limit of finite étale covers.
  (Iterate adjoining p-power roots of units.)
- Its tilt $\tilde{X}^\flat$ is affinoid and lives in equal characteristic, hence is a $K(\pi, 1)$ for p-torsion coefficients.
- As $\tilde{X}_{\text{ét}} \cong \tilde{X}^\flat_{\text{ét}}$, also $\tilde{X}$ is a $K(\pi, 1)$ for p-torsion coefficients.
- As $\tilde{X} \to X$ is pro-finite étale, also X is a $K(\pi, 1)$ for p-torsion coefficients.

$\square$