Singularities in formal arc spaces and harmonic analysis over non-archimedean fields

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- The nth jet space $J_nX(k) = X(k[t]/t^{n+1})$
- The formal arc space $\mathcal{L}X(k) = X(k[[t]])$ is the projective limit of the jets spaces.
- If X is affine, then $\mathcal{L}X$ is an infinite-dimensional algebraic variety.

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- In general, singularities of $\mathcal{L}X$ have finite dimensional formal models, after Drinfeld and Grinberg-Kazhdan.
- For every $x \in \mathcal{L}^\circ X$ a non-degenerate point of $\mathcal{L}X$, x maps the generic point of $\mathrm{Spec}(k[[t]])$ into the smooth part of X, then there exists a finite dimensional k-scheme Y with $y \in Y(k)$ such that $(\mathcal{L}X)_x \xrightarrow{\sim} Y_y \times \mathbb{D}^\infty$.

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- Question: are perverse sheaves on $\mathcal{L}^\circ X$ well defined?
- In particular, can the intersection complex of $\mathcal{L}^\circ X$ be defined?

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- In many instances, the IC-function have representation theoretic significance.

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- Typical example: $G = \mathrm{GL}_n$ and $M = \mathrm{Mat}_n$.

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- a representation $\rho : \hat{G} \to \mathrm{GL}(V_\rho)$ such that the restriction to $\mathbb{G}_m$ gives rise to the scalar multiplication on V_ρ.

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- $\psi = \sum_n \psi_n$ where ψ_n is a spherical Hecke function whose Satake transform is the trace of the nth symmetric power of ρ. The functions ψ_n have disjoint support.

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- the analytic continuation and functional equation of automorphic L-functions would follow.
- $\mathcal{S}_\rho(G(F))$ should be generated by trace function of perverse sheaves on $\mathcal{L}^\circ M_\rho$.

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- Instead of $\mathcal{L}^\circ M_\rho$, we can consider the space of maps $x : \mathbb{D} \to [M/G]$ mapping the generic point of of $\mathbb{D}$ on $[G/G]$.

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- $\mathcal{M}$ is invariant closed subscheme the Beilinson-Drinfeld Grassmannian. The calculation of the trace function on its IC follows the usual patters of convolution of equivariant sheaves on the affine Grassmannian.
- The global picture allows to compute ψ_n individually. The local geometry glues them together.

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Relative situation: loop geometry of invariant theory

- Maps from $\mathbb{D}$ to a $[M/G]$ containing the point as open subset.
- We may consider a situation $f : Y \to X$ where Y is an Artin stack, X is a scheme, f is generically an isomorphism.
- Want to study the space of maps $\mathbb{D} \to Y$ over a given map $\mathbb{D} \to X$.

The Jacquet-Rallis integral

- Let $G = \mathrm{GL}_n$, $\mathfrak{g} = \mathrm{Lie}(G)$, $H = \mathrm{GL}_{n-1}$ subgroup of G, acting on $\mathfrak{g}$ by the adjoint action.

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- Let $\mathcal{S}$ be the space of locally constant functions with compact support on on $\mathfrak{g}(F)$.
- For every $\phi \in \mathcal{S}$, we want to understand the function

$$x \mapsto \int_H \mathrm{ad}(h)x\mathrm{d}h$$

on the set of H-regular functions of $\mathfrak{g}(F)$.

Quotient stack

- consider the stack $\mathcal{Y}$ classifying quadruples $(V, x, v, v^\vee)$ where V is a n-dimensional vector space, $x \in \mathrm{End}(V)$, and $v \in V$, $v^\vee$.

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- $[\mathfrak{g}/H]$ is the closed substack of $\mathcal{Y}$ defined by the equation $\langle v^\vee, v\rangle = 1$.

Invariant theory

- The invariant theory gives rises to a morphism $f : \mathcal{Y} \to \mathfrak{b}$ with $\mathfrak{b} = \mathbb{A}^n \times \mathbb{A}^n$, defined by $(V, x, v, v^\vee) \mapsto (a, b)$ where $a_i = \mathrm{tr}(\wedge^i x)$ and $b_j = \langle v^\vee, x^j v \rangle$ for $i = 1, \ldots, n$ and $j = 0, \ldots, n-1$.

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- There exists a function $\gamma : \mathfrak{a} \times \mathfrak{b} \to \mathbb{G}_a$ such that the morphism $\mathcal{Y} \to \mathfrak{a} \times \mathfrak{b}$ is an isomorphism over the open subset $\mathfrak{b}^{\mathrm{reg}}$ defined by $\gamma \neq 0$.

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- $x \in \mathfrak{g}$ is H-regular if and only if $f(x) \in \mathfrak{b}^{\mathrm{reg}}$.

Asymptotics

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- The Jacquet-Rallis integral $x \mapsto \mathrm{JR}_x(\phi)$ defines a locally constant function on $\mathfrak{h}^{\mathrm{reg}}(F)$.
- Question: what is the asymptotic of this function as x tends to the boundary of $\mathfrak{h}^{\mathrm{reg}}$.

Basic function

- If $\phi = \mathbb{I}_{\mathfrak{g}(\mathcal{O})}$, then we expect that

$$\mathrm{JR}_x(\phi) = \sum_{i=0}^{n} \psi_i(x)$$

where ψ_i is the trace function of the closed subset of $\mathcal{L}^\circ \mathfrak{b}$ defined by $\mathrm{val}(\gamma) \geq i$.

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- Similar equality in the global setting is a theorem of Z. Yun.
- We expect for every $\phi \in \mathcal{S}$, the function $x \mapsto \mathrm{JR}_x(\phi)$ has asymptotic of the same form.

Adjoint action

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- The quotient map $[\mathfrak{g}/G] \to \mathfrak{a}$ is not an isomorphism because of the generic stabilizer.

Orbital integral

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- For $\phi \in C_c^\infty$, the orbital integral

$$O_x(\phi) = \int_{G(F)/G_x(F)} \phi(\mathrm{ad}(g)^{-1}x)\mathrm{d}g/\mathrm{d}t$$

depends on the choice of Haar measure $\mathrm{d}t$ on the centralizer $G_x(F)$.

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- It induces a function on $\Gamma_{\xi} : \mathcal{L}^{\circ}\mathfrak{a}(k) \to \mathbf{C}$.
- Question: Are these functions connected to perverse sheaves on $\mathcal{L}^{\circ}\mathfrak{a}$?

Local and global geometry

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- The stack of maps $a : \mathbb{D} \to [\mathfrak{g}/G]$ lying over a is not locally of finite type.
- If C is a smooth projective curve, and $\mathcal{T}$ a $\mathbb{G}_m$-torsor over C, then the stack of maps $a : C \to [\mathfrak{g}/G] \wedge^{\mathbb{G}_m} \mathcal{T}$ is locally of finite type. Example: the Hitchin fibration.