Roots of Unity in Intermediate Characteristic

July 24, 2015

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How do roots of unity behave in the intermediate regime?

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Warning

For $0 < n < \infty$, the fields $K(n)$ are not commutative.

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For $0 < n < \infty$, $L_{K(n)}$ is a mixture of “localization” and “completion.”

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- We will consider only *affine* schemes.
- There are more affine schemes than you might think.

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Question

What does "$S = \operatorname{Spec} \mathbb{C}^S$" mean?

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$$(s \in S) \mapsto \mathfrak{p}_s = \{f : S \to \mathbb{C} | f(s) = 0\}$$

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$$\{ \text{ local systems of vector spaces on } S \ \} \simeq \{ \ \mathbb{C}^S\text{-modules } \}$$

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Theorem (Hopkins, L)

If X is a p-finite n-truncated space, then the global sections functor

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Example

$$\mu_p := \operatorname{Spec} R[\mathbb{Z}/p\mathbb{Z}].$$

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Question

When is $\operatorname{Spec} R^X$ a sheafification of $\underline{X}$?

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Condition (a) is very strong!

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Example

Let G be a finite p-group. Then BG is affine.

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$A \simeq W(k)[[v_1, \ldots, v_{n-1}]]$.

Theorem (Landweber, Morava, Goerss-Hopkins-Miller)

There is an essentially unique even periodic cohomology theory E with $E^0(*) = A$ and $\mathbb{G} = \operatorname{Spf} E^0(\mathbb{CP}^\infty)$.

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Slogan

$K(n)$ is the "residue field" of E.

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$$\pi_* E = \begin{cases} W(k)[[v_1, \ldots, v_{n-1}]]^\times & \text{if } * = 0 \\ 0 & \text{if } * \text{ is odd} \\ W(k)[[v_1, \ldots, v_{n-1}]] & \text{if } * \geqslant 2 \text{ is even.} \end{cases}$$

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Marble	$\mathbb{C}^\times$	$E^\times$
Chisel	μ_p	μ_p
Angel	$\mu_{p^\infty}(\mathbb{C}) \simeq \mathbb{Q}_p/\mathbb{Z}_p$	???

Excavating the Interesting Part of $E^\times$

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Corollary

$$\pi_* X = \begin{cases} \mathrm{Hom}(\pi^s_{n-*}, \mathbb{Q}_p/\mathbb{Z}_p) & \text{if } 0 \leqslant * \leqslant n \\ 0 & \text{if } * > n. \end{cases}$$