Representation Theory in Intermediate Characteristic

July 21, 2015

In Algebra (p-locally)

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Characteristic Zero	Characteristic p
$\mathbf{Q}$	$\mathbf{F}_p$

In Homotopy Theory (p-locally)

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Characteristic Zero	Morava K-Theories	Characteristic p

$$H\mathbf{Q} = K(0) \quad K(1) \quad K(2) \quad K(3) \quad \cdots \quad K(\infty) = H\mathbf{F}_p$$

In Homotopy Theory (p-locally)

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Question

What happens to the representation theory of finite groups over these intermediate fields?

Representation Theory of Finite Groups (Classical)

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Study vector spaces V over k with an action $G \circlearrowright V$.

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Phenomenon (Complete Reducibility)

For any representation V, the subspace of invariant vectors

$$V^G = \{v \in V : (\forall g \in G)[gv = v]\}$$

is a direct summand of V.

Proof of Complete Reducibility (Characteristic Zero)

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Phenomenon (Norm Isomorphisms)

For any representation V, the norm map

$$N_G : V_G \to V^G$$

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Phenomenon (Unipotence)

Every representation of G can be built as a successive extension of trivial representations.

Complete Reducibility vs. Unipotence

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Question

What happens in between?

Local Systems of Vector Spaces

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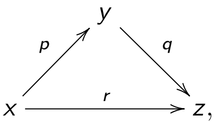

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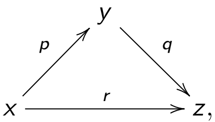

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For X connected,

$$\text{local systems on } X \;\simeq\; \text{representations of } \pi_1 X$$

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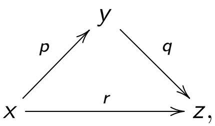

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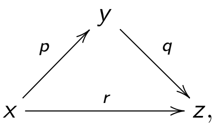

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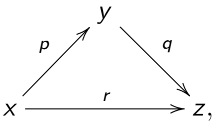

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Warning (or Feature)

This does not depend only on $\pi_1 X$.

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We say X is p-*finite* if it is π-finite and each $\pi_n(X, x)$ is a p-group.

Example

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local systems on BG $\simeq$ "representations of G"

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Phenomenon: Norm Isomorphisms

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In particular, $K(n)^0 X$ is finite-dimensional and self-dual.

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The bilinear b gives an identification of $K(1)^0 BG$ with its dual (over $\mathbf{F}_p$). This agrees with the duality map of the previous slide.

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Corollary

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Corollary

If X is π-finite, then the construction $\mathcal{L} \mapsto C^*(X; \mathcal{L})$ commutes with direct limits. Likewise, $\mathcal{L} \mapsto C_*(X; \mathcal{L})$ commutes with inverse limits.

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Idea of proof: take $\mathcal{L}^{\mathrm{unip}}$ to be the direct limit of all constant local systems with a map to $\mathcal{L}$.

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Example

If G is a finite p-group, then every representation of G (on a $K(n)$-module) is unipotent.

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Remark

If $\pi_m X \simeq 0$ for $m \leqslant n+1$, then any local system of $K(n)$-modules on X is constant.

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Slogan

The larger n is, the more unipotence we see.

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The larger n is, the more unipotence we see. (Because we are getting closer to characteristic p.)