Representation Theory in Intermediate Characteristic

July 21, 2015
In Algebra ($p$-locally)
In Algebra \((p\text{-locally})\)

<table>
<thead>
<tr>
<th>Characteristic Zero</th>
<th>Characteristic (p)</th>
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<tr>
<td>( \mathbb{Q} )</td>
<td>( \mathbb{F}_p )</td>
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In Homotopy Theory ($p$-locally)
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<td>(HQ) = (K(0))</td>
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In Homotopy Theory ($p$-locally)

Characteristic Zero  Morava $K$-Theories  Characteristic $p$

$$HQ = K(0) \quad K(1) \quad K(2) \quad K(3) \quad \cdots \quad K(\infty) = HF_p$$

**Question**

What happens to the representation theory of finite groups over these intermediate fields?
Representation Theory of Finite Groups (Classical)

$G$ is a finite group.

$k$ is a field.

Study vector spaces $V$ over $k$ with an action $G \triangleright V$. 
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Study vector spaces $V$ over $k$ with an action $G \curvearrowright V$. 
Representation Theory in Characteristic Zero

In characteristic zero we have complete reducibility:
Every representation $V$ is a direct sum of irreducible representations.
Every exact sequence of representations $0 \rightarrow V_1 \rightarrow V \rightarrow V_2 \rightarrow 0$ splits.

Phenomenon (Complete Reducibility)
For any representation $V$, the subspace of invariant vectors $V_G = \{ v \in V : g \cdot v = v \}$ is a direct summand of $V$. 
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$$0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$$

splits.

**Phenomenon (Complete Reducibility)**

*For any representation $V$, the subspace of invariant vectors*

$$V^G = \{ v \in V : (\forall g \in G)[gv = v] \}$$

*is a direct summand of $V$.***
Proof of Complete Reducibility (Characteristic Zero)
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Define $V_G = V/(gv - v)$. 
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$V^G \hookrightarrow V \twoheadrightarrow V_G$
Proof of Complete Reducibility (Characteristic Zero)

Define \( V_G = V / (gv - v) \).

\[ V^G \leftrightarrow V \rightarrow V_G \]

The construction \( v \mapsto \sum_{g \in G} gv \) factors

\[ V \rightarrow V_G \xrightarrow{N_G} V^G \leftarrow V \]
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The composition in either direction is multiplication by $|G| \neq 0$. 
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$V \twoheadrightarrow V_G \xrightarrow{N_G} V^G \hookrightarrow V$

The composition in either direction is multiplication by $|G| \neq 0$.

Phenomenon (Norm Isomorphisms)

*For any representation $V$, the norm map*

$N_G : V_G \rightarrow V^G$

*is an isomorphism.*
Representation Theory in Characteristic $p$

Complete reducibility fails if $|G| > 0$ in $k$. If $G$ is a finite $p$-group for $p > \text{char } k$, we instead get unipotence:

Every irreducible representation of $G$ is trivial.

Phenomenon (Unipotence)

Every representation of $G$ can be built as a successive extension of trivial representations.
Representation Theory in Characteristic $p$

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Complete Reducibility vs. Unipotence
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**Question**

What happens in between?
Complete Reducibility vs. Unipotence

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What happens in between?
Local Systems of Vector Spaces

A local system $L$ of $k$-vector spaces on a space $X$ assigns:

To each point $x \in X$ a $k$-vector space $L_x$.

To each path $p: r \to s \to X$ from $x \to y$ a map $L_p: L_x \to L_y$.

For each 2-simplex $\Delta: r \to s \to t \to X$, we have $L_r \circ L_p = L_t$.

For $X$ connected, local systems on $X$ correspond to representations of $\pi_1(X)$. 
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- For each 2-simplex \( \Delta^2 \to X \)

\[
\begin{array}{ccc}
& y & \\
p & \downarrow & q \\
\downarrow & & \downarrow \\
X & \rightarrow & z,
\end{array}
\]

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For \( X \) connected, local systems on \( X \) \( \cong \) representations of \( \pi_1 X \).
Local Systems of $K(n)$-Modules
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A local system $\mathcal{L}$ of $K(n)$-modules on a space $X$ assigns:

- To each point $x \in X$ a $K(n)$-module spectrum $L_x$.
- To each path $p_0, 1 \to x \to y$ a homotopy equivalence $L_p : L_x \to L_y$.
- For each 2-simplex $\Delta_2 \to X$ a homotopy $L_r \to L_q \circ L_p$.

Analogous data for simplices of all dimensions.

Warning (or Feature) This does not depend only on $\pi_1 X$. 
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\[
\begin{array}{c}
y \\
\downarrow p \\
\downarrow q \\
\leftarrow r \\
\leftarrow z, \\
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Finiteness Hypotheses

Definition
A space $X$ is $\pi$-finite if:

- The set $\pi_0 X$ is finite.
- Each homotopy group $\pi_n X$, $x_0$ is finite.
- The groups $\pi_n X$, $x_0$ vanish for $n \neq 0$.

We say $X$ is $p$-finite if it is $\pi$-finite and each $\pi_n X$, $x_q$ is a $p$-group.
Finiteness Hypotheses

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We say $X$ is $p$-finite if it is $\pi$-finite and each $\pi_n(X, x)$ is a $p$-group.
Example

A finite group $G$ is $\pi$-finite if $BG$ is $\pi$-finite, and a finite $p$-group $G$ is $p$-finite if $BG$ is $p$-finite. Local systems on $BG$ correspond to representations of $G$. 
Example

\( G \) a finite group \( \Rightarrow BG \) is \( \pi \)-finite
Example

$G$ a finite group $\Rightarrow BG$ is $\pi$-finite

$G$ a finite $p$-group $\Rightarrow BG$ is $p$-finite
Example

\[ G \text{ a finite group} \implies BG \text{ is } \pi\text{-finite} \]

\[ G \text{ a finite } p\text{-group} \implies BG \text{ is } p\text{-finite} \]

local systems on BG \(\simeq\) “representations of \(G\)”
Invariants and Coinvariants
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# Invariants and Coinvariants

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Phenomenon: Norm Isomorphisms
Theorem (Hopkins, L)

Let $X$ be a $\pi$-finite space and $\mathcal{L}$ a local system of $K(n)$-modules on $X$. There is a canonical norm isomorphism $N_X^L : C_\pi^\wedge X; \mathcal{L} \to C_\pi^\wedge X; \mathcal{L}$. Example (Duality) If $\mathcal{L}$ is the trivial local system, then $K^\wedge X \to N K^\wedge X$. In particular, $K^\wedge 0 X$ is finite-dimensional and self-dual.
Phenomenon: Norm Isomorphisms

**Theorem (Hopkins, L)**

Let $X$ be a $\pi$-finite space and $\mathcal{L}$ a local system of $K(n)$-modules on $X$. There is a canonical norm isomorphism

$$N_X : C_{\ast}(X; \mathcal{L}) \xrightarrow{\sim} C^{\ast}(X; \mathcal{L}).$$
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**Example (Duality)**

If $\mathcal{L}$ is the trivial local system, then

$$K(n)_*X \xrightarrow{\sim} K(n)^*X.$$
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In particular, $K(n)^0 X$ is finite-dimensional and self-dual.
Example: $K(1) = K/p$

Let $G$ be a finite $p$-group and let $\text{Rep}(G)$ be its representation ring.
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Let $G$ be a finite $p$-group and let $\text{Rep}(G)$ be its representation ring. There is a nondegenerate bilinear form

$$b : \text{Rep}(G) \otimes \text{Rep}(G) \to \mathbb{Z}$$

$$(V, W) \mapsto \dim_{\mathbb{C}} \text{Hom}(V, W)$$
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$$b : \text{Rep}(G) \otimes \text{Rep}(G) \rightarrow \mathbb{Z}$$

$$(V, W) \mapsto \dim_{\mathbb{C}} \text{Hom}(V, W) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)} \chi_W(g)$$

The Atiyah-Segal completion theorem gives an isomorphism $K_{p1} \cong \text{Rep}(G)_{\mathbb{F}_p}$. The bilinear $b$ gives an identification of $K_{p1} \cong \text{Rep}(G)_{\mathbb{F}_p}$ with its dual (over $\mathbb{F}_p$). This agrees with the duality map of the previous slide.
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The bilinear $b$ gives an identification of $K(1)^0BG$ with its dual (over $\mathbb{F}_p$). This agrees with the duality map of the previous slide.
Another Corollary

Let $X$ be any space. The construction $L \Rightarrow C \Rightarrow p X \Rightarrow L_q$ commutes with inverse limits. But usually not with direct limits.

Likewise, $L \Rightarrow C \Rightarrow p X \Rightarrow L_q$ commutes with direct limits.
Another Corollary

Let $X$ be any space.
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Let $X$ be any space.

- The construction $\mathcal{L} \mapsto C^*(X; \mathcal{L}) = \lim_{x \in X} \mathcal{L}_x$ commutes with inverse limits.
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- The construction $\mathcal{L} \mapsto C_*(X; \mathcal{L}) = \lim_{x \in X} \mathcal{L}_x$ commutes with direct limits.
Another Corollary

Let $X$ be any space.

- The construction $\mathcal{L} \mapsto \mathcal{C}^*(X; \mathcal{L}) = \lim_{x \in X} \mathcal{L}_x$ commutes with inverse limits. But usually not with direct limits.

- The construction $\mathcal{L} \mapsto \mathcal{C}_*(X; \mathcal{L}) = \lim_{x \in X} \mathcal{L}_x$ commutes with direct limits. But usually not with inverse limits.
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**Corollary**

*If $X$ is $\pi$-finite, then the construction $\mathcal{L} \mapsto \mathcal{C}^*(X; \mathcal{L})$ commutes with direct limits.*
Another Corollary

Let $X$ be any space.

- The construction $\mathcal{L} \mapsto C^*(X; \mathcal{L}) = \lim_{x \in X} \mathcal{L}_x$ commutes with inverse limits. But usually not with direct limits.
- The construction $\mathcal{L} \mapsto C_*(X; \mathcal{L}) = \lim_{x \in X} \mathcal{L}_x$ commutes with direct limits. But usually not with inverse limits.

**Corollary**

*If $X$ is $\pi$-finite, then the construction $\mathcal{L} \mapsto C^*(X; \mathcal{L})$ commutes with direct limits. Likewise, $\mathcal{L} \mapsto C_*(X; \mathcal{L})$ commutes with inverse limits.*
Application: Unipotent Local Systems
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**Definition**

A local system $\mathcal{L}$ of $K(n)$-modules on a space $X$ is *unipotent* if it can be built from constant local systems using direct limits.
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Corollary (of Theorem)

If $X$ is $\pi$-finite, then every local system $\mathcal{L}$ can be written as an extension

$$\mathcal{L}^{\text{unip}} \to \mathcal{L} \to T$$

where $\mathcal{L}^{\text{unip}}$ is unipotent and $C^*(X; T) \cong 0$. 
Application: Unipotent Local Systems

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where $\mathcal{L}^{\text{unip}}$ is unipotent and $C^*(X;\mathcal{T}) \simeq 0$.

Idea of proof: take $\mathcal{L}^{\text{unip}}$ to be the direct limit of all constant local systems with a map to $\mathcal{L}$. 
Phenomenon: Unipotence
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**Theorem (Hopkins, L)**

Let $X$ be a $p$-finite space and assume that $\pi_m X \simeq 0$ for $m > n$. 
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**Example**

If $G$ is a finite $p$-group, then every representation of $G$ (on a $K(n)$-module) is unipotent.
Representation Theory in Intermediate Characteristic

Phenomenon: Complete Reducibility

Let $X$ be a $\pi$-finite space. Assume $|\pi^m X|$ is not divisible by $p$ for $m \leq n$.

Then:

1. Every unipotent local system on $X$ is constant.
2. For every local system $L$ on $X$, the extension $L^{\text{unip}} - L^{\text{K}}$ splits.

Remark: If $\pi^m X \rightarrow 0$ for $m \leq n < 1$, then any local system of $K_p^n$-modules on $X$ is constant.
Phenomenon: Complete Reducibility

Theorem

Let $X$ be a $\pi$-finite space.
Phenomenon: Complete Reducibility

**Theorem**

Let $X$ be a $\pi$-finite space. Assume $|\pi_m X|$ is not divisible by $p$ for $m \leq n$. Then:

Every unipotent local system on $X$ is constant.
For every local system $L$ on $X$, the extension $L_{\text{unip}} \rightarrow L_{\text{K}}$ splits.

Remark: If $\pi_m X \twoheadrightarrow 0$ for $m \leq n$, then any local system of $K_p n_q$-modules on $X$ is constant.
Phenomenon: Complete Reducibility

**Theorem**

Let $X$ be a $\pi$-finite space. Assume $|\pi_m X|$ is not divisible by $p$ for $m \leq n$. Then:

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**Theorem**

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**Theorem**

Let $X$ be a $\pi$-finite space. Assume $|\pi_m X|$ is not divisible by $p$ for $m \leq n$. Then:

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- For every local system $\mathcal{L}$ on $X$, the extension

$$\mathcal{L}^{\text{unip}} \rightarrow \mathcal{L} \rightarrow \mathcal{K}$$

splits.

**Remark**

If $\pi_m X \simeq 0$ for $m \leq n + 1$, then any local system of $K(n)$-modules on $X$ is constant.
Representation Theory in Intermediate Characteristic

For local systems of $K_{p^n}$-modules on a $p$-finite space $X$:

Unipotence

$$\pi_0 X \pi_1 X \cdots \pi_n X$$

Complete Reducibility

$$\pi_n \cdots \pi_1 X$$

Triviality

$$\pi_n \cdots \pi_2 X$$

Slogan

The larger $n$ is, the more unipotence we see. (Because we are getting closer to characteristic $p$.)
For local systems of $K(n)$-modules on a $p$-finite space $X$:
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Unipotence

\[
\left\{
\begin{array}{l}
\pi_0 X \\
\pi_1 X \\
\ldots \\
\pi_n X
\end{array}
\right.
\]

(Slogan) The larger $n$ is, the more unipotence we see. (Because we are getting closer to characteristic $p$.)
For local systems of $K(n)$-modules on a $p$-finite space $X$:

- **Unipotence**: \( \pi_0 X, \pi_1 X, \ldots, \pi_n X \)
- **Complete Reducibility**: \( \pi_{n+1} X \)

**Slogan**: The larger $n$ is, the more unipotence we see. (Because we are getting closer to characteristic $p$.)
For local systems of $K(n)$-modules on a $p$-finite space $X$:

**Unipotence**

\[
\begin{align*}
\pi_0X \\
\pi_1X \\
\ldots \\
\pi_nX
\end{align*}
\]

**Complete Reducibility**

\[
\begin{align*}
\pi_{n+1}X
\end{align*}
\]

**Triviality**

\[
\begin{align*}
\pi_{n+2}X \\
\ldots
\end{align*}
\]

Slogan: The larger $n$ is, the more unipotence we see. (Because we are getting closer to characteristic $p$.)
For local systems of $K(n)$-modules on a $p$-finite space $X$:

Unipotence

\[
\begin{cases}
\pi_0 X \\
\pi_1 X \\
\vdots \\
\pi_n X
\end{cases}
\]

Complete Reducibility

\[
\begin{cases}
\pi_{n+1} X \\
\pi_{n+2} X \\
\vdots
\end{cases}
\]

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For local systems of $K(n)$-modules on a $p$-finite space $X$:

\begin{align*}
\text{Unipotence} & \quad \left\{ \begin{array}{l}
\pi_0 X \\
\pi_1 X \\
\ldots \\
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\end{array} \right.
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\pi_{n+2} X \\
\ldots \\
\end{array} \right.
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\end{align*}

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