Cohomology Theories and Commutative Rings

July 20, 2015

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Let us (temporarily) fix the group A and denote $\mathrm{H}^n(X; A)$ by $\mathrm{H}^n(X)$.

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(*A*6) Dimension Axiom: $\mathrm{H}^n(*) = \begin{cases} A & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$

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Any collection of invariants $\{\mathrm{H}^n : \text{ spaces } \to \text{ abelian groups } \}$ *satisfying* $(A1)$ *through* $(A6)$ *is given by cohomology with coefficients in the abelian group* $A = \mathrm{H}^0(*)$.

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A *cohomology theory* E is a sequence of invariants $\{E^n : \text{ spaces } \to \text{ abelian groups}\}$ satisfying $(A1)$ through $(A5)$. (The isomorphisms $E^n_{\text{red}}(X) \simeq E^{n+1}_{\text{red}}(\Sigma X)$ are part of the data.)

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$$K^n(*) = \begin{cases} \mathbb{Z} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

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The suspension isomorphisms $E^n_{\text{red}}(X) \simeq E^{n+1}_{\text{red}}(\Sigma X)$ determine homotopy equivalences $Z(n) \simeq \Omega Z(n+1)$.

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Using Brown's theorem, one gets a bijection

$$\{\text{cohomology theories}\}/\text{iso} \simeq \{\text{spectra}\}/\text{homotopy equivalence.}$$

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The sequence $\{K(A, n)\}$ determines a spectrum denoted by HA, the *Eilenberg-MacLane spectrum* of A.

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There is a similar notion of *commutative (or E_∞) ring spectrum*.

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Complex K-theory is represented by a commutative ring spectrum K.

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So K/p is a cohomological field.

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This is an equivalence relation $\sim$.

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- If k is a field of characteristic p, then $\mathrm{H}^n(X; k) \simeq k$ for all $n \geq 0$.
- $(K/p)^n(X) = \begin{cases} \mathrm{Rep}(G) \otimes (\mathbb{Z}/p\mathbb{Z}) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$

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Corollary

K/p is not of the same characteristic as any ordinary field.

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- If not, one can show that the rank of $E^*(X)$ over $E^*(\{x\})$ is p^n for $0 < n < \infty$. We will say that E has *height* n.

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Example

We can take $K(1) = K/p$.

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In Lecture 2, we'll discuss the behavior of representation theory over these fields of "intermediate characteristic".

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$$K(1) \qquad K(2) \qquad K(3) \qquad \cdots$$

These interpolate between fields of characteristic zero ($K(0) \simeq H\mathbf{Q}$) and fields of characteristic p ($K(\infty) \simeq H\mathbf{F}_p$).

In Lecture 2, we'll discuss the behavior of representation theory over these fields of "intermediate characteristic".

In Lecture 3, we'll study some rudimentary algebraic geometry in these settings, focusing in particular on roots of unity.