Algebraic Geometry: countably many fin.-dim. families of algebraic varieties

Symplectic Topology: countably many fin.-dim. families of triangulated $A_{\infty}$-categories

"non-commutative rigid analytic spaces endowed with a section of anti-canonical bundle" (pre-Calabi-Yau)
moduli spaces are schemes (stacks) of finite type $/Spec \mathbb{Z}$ usually singular.

parameter spaces are smooth formal schemes $/Spec \mathbb{Z}$ - divisors with canonical flat coordinates $Spec \mathbb{Z} ((q_1, \ldots, q_r ))$

$r = rk H^2 (\text{symplectic manifold})$
Basic building blocks in ST:

Lagrangian skeleta

Singular isotropic subsets

Weinstein "manifolds"

Example: \( X \subset \mathbb{C}^N \) affine

\[ \omega = \frac{1}{2} \sum_{j=1}^N d\bar{z}_j \wedge d\bar{z}_j \mid_X \]

\[ \xi = \text{grad} \left( \sum_{j=1}^N |z_j|^2 \right) \mid_X \]
Compact Lagrangian skeleton $L = X$

$\rightsquigarrow$ Triangulated $\infty$-category $\mathcal{F}(L)/\mathbb{Z}$

$\text{dg}$

\[ \omega_x = d(-i_2^* \omega) \]

$\left\{ \omega_x \right\} = 0$

Analog of $\mathcal{D}^b(\text{Coh } Y)$

where $Y/\mathbb{Z}$ is a scheme with Gorenstein singularities

+ trivialized canonical bundle

$K_Y \simeq \mathcal{O}_Y$

Idea of definition:

$\mathcal{F}(L) = \text{Perf } A_L\text{-mod}$

$A_L := \text{End}_{\text{wrapped Fukaya}}$ (transversal disks)

\[ \text{base points in } L - \text{sing} \]
Noncompact version

Example: \( L = \mathbb{R} \)

\[ A_L = \mathbb{Z} \]

\[ \mathcal{F}(L) = \text{complexes} \]

In general \( L = \overline{I} - \mathcal{E}_\infty L \)

Analog of mfd with boundary

Compact singular lagrangian

Collar:

\[ \mathbb{E} \times [0,3) \]
Case: $\dim L = 1$

Ribbon graph

Functors $F(L) \to \mathcal{A}$ category $\mathcal{C} = \text{A edge } e \in L \mapsto \text{object } \mathcal{C}_{e \in \mathcal{C}}$

+ A vertex of degree $d \geq 1$: representation of quiver $A_{d-1}$ in $\mathcal{C}$

- $d = 1$
  \[ \begin{array}{c}
  \vdots \\
  \mathcal{C}_1 \\
  \vdots \\
  \end{array} \] : $\mathcal{C}_1 = 0$

- $d = 2$
  \[ \begin{array}{c}
  \mathcal{C}_1 \\
  \mathcal{C}_2 \\
  \vdots \\
  \end{array} \] : $\mathcal{C}_2 = 0$

- $d = 3$
  \[ \begin{array}{c}
  \mathcal{C}_1 \\
  \mathcal{C}_2 \\
  \mathcal{C}_3 \\
  \vdots \\
  \end{array} \] : exact triangle $\mathcal{C}_{1,2} \to \mathcal{C}_3$
\[ F(\cdot) = F(\mathbb{R}) = F(\mathbb{R}^2) = \cdots \sim D^b(Coh(\cdot)) \]

\[ S^1, (d\tau/\tau)^2 \sim Spec \mathbb{Z}[t, t^{-1}] = \mathbb{A}^1 - 0 \]

\[ \sim Spec \mathbb{Z}[t] = \mathbb{A}^1 \]

\[ \sim \mathbb{P}^1 \]
More exotic:

\[ \sim \text{Spec } \mathbb{Z}[x,y]/xy=0 = \frac{\mathbb{A}^1}{\mathbb{A}^1} \sim \text{singular} \]

\[ \bigcirc \sim \text{Spec } \mathbb{Z}[x]/x^2=0 \sim \text{nilpotent} \]

\[ N\{\text{something}\} \sim \mathbb{P}^1/\mathbb{G}_m \sim \mathbb{P}^1/\mathbb{G}_m \text{ Deligne-Mumford stack} \]

\[ \sim \text{Artin stack} \]
$\dim = 2$ Torus $\cup$ 2 tubes $\sim A^2 - 0$

$S' \times S' \cup (\text{Disc}^2 \times pt)$

$\sim \text{Spec } \mathbb{Z} [x,y,z] / (xy - 1)z = 1$

simplest non-toric cluster variety

$\dim = 3$

$\text{Cone} / (S')^2$

$\sim \mathbb{P}^1 - \{0, 1, \infty\}$
David Nadler: Can deform $L$ (like $X \rightarrow \mathcal{M}$) to one with "simple" singularities
\[ \Rightarrow \text{ explicit combinatorial description of } \mathcal{F}(L). \]

Important feature: $(L' \subset \text{ open } L) \Rightarrow \mathcal{F}(L')$ is a cosheaf.

If $L$ is smooth $\sim K(\Gamma, 1) \Rightarrow \mathcal{F}(L) = \text{Perf } (\mathbb{Z}[\Gamma]\text{-mod})$
(possibly noncompact)

\[ \Rightarrow \text{ twisted by } u_2 \]
Stiefel–Whitney $\in H^2(L, \mathbb{Z}_2)$
New improved definition of Fukaya category:

Given $(X,\omega)$ compact symplectic manifold

+ trivialization $2c_1(Tx) = 0$

+ some data at $\infty$ if $X$ is not compact:
  convexity, partial wrapping

$\implies$ triangulated $A_\infty$-category $\mathcal{F}(X,\omega)$

over Novikov field $\mathbb{Q}(\mathbb{R}((q^1, \ldots, q^n))) = \left\{ \sum a_i q^i \mid a_i \in \mathbb{Q}, E_i \in \mathbb{R}, \lim_\infty E_i = +\infty \right\}$
A singular Lagrangian $L \subset X$ at $\infty$ $L$ goes to the allowed part $\forall \infty X$

$\Rightarrow$ Formal deformation of $F(L)$

$/ Q[[q^{R>0}]]$

Should be considered as a black box

Governed by pseudo-holomorphic discs $D^2 \subset X, \partial D^2 \subset L$

Contribute $\sum_{Area>0} q^{-\infty}$

**Definition**: $F(X,\omega) = \lim (\text{Functors } (F(L) \text{ deformed}), D^b(\text{pt}))$

Fully faithful embeddings

over all $L \subset X$

$\bigcirc \subset \bigodot \subset \bigcirc \subset \bigodot \subset \ldots$
Rules of the game:

- If by some geometric reasons for some almost-complex structure $J$ on $(X,\omega)$ we know that there are non-trivial $J$-holomorphic discs with boundary on $L$, then we have a fully faithful embedding $\mathcal{F}(L) \hookrightarrow \mathcal{F}(X,\omega)$.

  Warning: different reasons can give different embeddings.

- If $L$ is smooth and $\sigma : (X,\omega) \to (X,\omega)$ is an antisymplectic involution with $\sigma^* \omega = -\omega$, then $\sigma|_L = \text{id}$ implies that a trivial local system on $L$ gives an object $[L,\sigma] \in \mathcal{F}(X,\omega)$. Like $w = \omega_L = df$. 

Example: \( L \leq X \) Lagrangian torus \((\text{M. Abouzaid})\)

\[ L \simeq (S^1)^n \simeq K(Z^n,1) \]

\( \Rightarrow \) get deformation of \( \mathbb{Z}[x_1^\pm,\ldots,x_n^\pm] \)

as (possibly) a **non-commutative** algebra

Could be \( \Rightarrow \) truly non-commutative (deformation quantization)

\( \uparrow \)

\{ can be ruled out by involution, purely commutative, hence \( \nu \) trivial (not canonically) \}

Area-length inequality

\[ \text{Area} (\text{Disc}) \geq \text{const} \cdot \text{Length}(\partial \text{Disc}) \sim \text{in non-archimedean} \]

\[ \Rightarrow \text{open domain } U_L \text{ in } \mathbb{G}_m^n \text{ an} \]
Situation of SYZ-fibration

\[ X \rightarrow B \]
\[ \dim_{\mathbb{R}} B = h = \frac{1}{2} \dim_{\mathbb{R}} X \]

Can glue finitely many domains \( U_{L,i} \) (also for singular fibers)

and obtain Mirror Dual

which is rigid-analytic manifold \( X^r \)

\[ \mathcal{F}(X, \omega) = \text{Perf} (X^r) \]

could be very small if \( X^r \) is not algebraic
In general, not in situation of Mirror Symmetry
Area-length inequality $\Rightarrow$ Finitely many $L_i \subset \mathcal{X}$
will suffice.

"Noncommutative open analytic domains"

$$U_{\mathcal{L}_i} \supset \widetilde{U}_{\mathcal{L}_i}$$
relatively compact

$$\bigcup U_{\mathcal{L}_i} = \bigcup \widetilde{U}_{\mathcal{L}_i}$$
à la Grauert
PART II.

Mirror duals in non-archimedean geometry.
(work in progress with Tony Yue Yu)

\((X, \text{vol})\) possibly noncompact analytic Calabi-Yau

/ non-archimedean field \(K\)

\(\lim X^n \rightarrow \text{essential skeleton} \)

\(\text{Ske}_e(X)\) finite PL complex

\[\text{Def.: Maximal degeneration if} \]

\[\dim_{\mathbb{R}} \text{Ske}_e(X) = \dim X/K\]
E.g. if $X/K$ is algebraic with model $\bar{X}/O_K$ regular

st. special fiber $X_0$ is divisor with normal crossings. $= U D_i$

$\longrightarrow S_k \in (X_{ud}) \subset$ Clemens complex

subcomplex

(vert.) $\leftrightarrow D_i$

(faces) $\leftrightarrow D_{ij \ldots nD_i \neq 0}$

Spanned by $\{D_i\}$ such that vol has maximal order at pole.
A maximally irrational point \( p \in \text{Sk}_{es}(X, \text{vol}) \)

\[
\sum a_i t_i \neq 0 \quad \forall (a_0, \ldots, a_n) \in \mathbb{Z}_{-0}^n
\]

\( n \)-simplex

\( n = \dim X \)

\( p : t_0 + \cdots + t_n = 1 \)

\[
3 \text{ canonical germ of analytic tube domain } \rightarrow (G_m^n)_{an} \subset X
\]

near \( S^1_{t_0} \times \cdots \times S^1_{t_n} = L_p \)

\( S^1_t := \{ z \in G_m^n \mid |z| = t \} \) circle of radius \( t \).

+ canonical \( \mathbb{Z} \)-affine structure on \( \text{Sk}_{es}(X, \text{vol}) \) near \( p \).
Claim: \( \forall p \in Sk_{\text{os}}(X, \text{vol}) \)

\[ \rightarrow \text{Countable set } T_p^\mathbb{Z} \quad (\mathbb{Z}-\text{points in tangent cone } T_p) \]

(Typically \( \sim \mathbb{Z}^n \)

up to \( \mathbb{Z} \) piecewise-linear maps)

\[ + \quad \underline{\text{Multiplication Table}} \quad \underline{\text{Basis } \theta_v, \, v \in T_p^\mathbb{Z}} \]

\[ \theta_v \cdot \theta_{v_2} = \sum_{v_3} c_{v,v_2,v_3} \theta_{v_3} \quad v_i \in T_p^\mathbb{Z} \]

\[ c_{v,v_2} \in \mathbb{Z}[[X]] (\text{ample cone of } X) \]

? \( \mathbb{Z}[\frac{1}{q}] \) if \( \text{char (residue field)} = \ell > 0 \)
If $p$: maximally irrational point $\Rightarrow$ basis of monomials in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

Otherwise: generalization of canonical basis in cluster algebras.

**Formula:** $C_{v_1 v_2 v_3} = \text{genus}=0 \text{ Gromov-Witten invariant}$

in an auxiliary space $X \cup 3$ flaps near $p$

Should work in $A$ characteristic and $e \in \mathbb{Z}$ if $\text{char}=0$

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Choose non-archimedean Kähler metric on $X$

by analogs of area-length inequalities

get open domains $U_p \subset X$

(analogous to $U_{L_p}$ from the previous story,)

Think $L_p = \text{torus over } p$.

**Conclusion:** In non-archimedean situation (especially in positive characteristic)

$\exists$ Fukaya category

but still $\exists$ a mirror!