Integral models of Shimura varieties II

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Main Theorem

Let (G, X) be a Shimura datum of abelian type.

$$\mathsf{K}_p = G_{\mathbb{Z}_p}(\mathbb{Z}_p) \subset G(\mathbb{Q}_p) \text{ hyperspecial}, \quad \mathsf{K}^p \subset G(\mathbb{A}_f^p) \text{ compact open}$$

$\mathsf{K} = \mathsf{K}_p\mathsf{K}^p \subset G(\mathbb{A}_f)$.

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Theorem. *Let $p > 2$, and $\lambda|p$ be a prime of E. There is a smooth $\mathcal{O}_{E_\lambda}$-scheme $\mathscr{S}_{\mathsf{K}}(G, X)$ extending $\mathrm{Sh}_{\mathsf{K}}(G, X)$ such that the $G(\mathbb{A}_f^p)$-action on*

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$$\mathscr{S}_{\mathsf{K}_p}(G, X)(R) = \mathscr{S}_{\mathsf{K}_p}(G, X)(R[1/p]).$$

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$\mathsf{K}'_p \subset \mathrm{GSp}(V_{\mathbb{Q}_p})$, the stabilizer of a lattice $V_{\mathbb{Z}_p} \subset V_{\mathbb{Q}_p}$, and

$$G_{\mathbb{Z}_p} \hookrightarrow \mathrm{GL}(V_{\mathbb{Z}_p}) \quad \text{(Prasad-Yu)}.$$

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Idea: Describe the local structure of $\mathscr{S}_{\mathsf{K}_p}(G,X)$ in terms of moduli of "p-adic Hodge structures", or more precisely p-divisible groups.

Hodge cycles.

Let $E(G,X) \subset K \subset \mathbb{C}$ be a field, and

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Theorem. *(Deligne: "Hodge $\Longrightarrow$ Absolute Hodge")*

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Now suppose K/E_λ is finite, with residue field of k.

Suppose $\mathcal{A}_x$ has good reduction:

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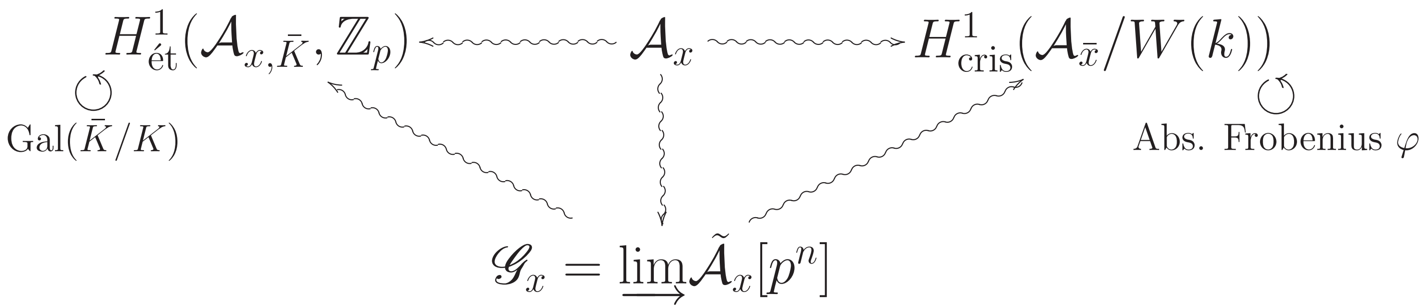

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B_{cris} is a $K_0 = W(k)[1/p]$-algebra with an action of $\mathrm{Gal}(\bar{K}/K)$ and φ.

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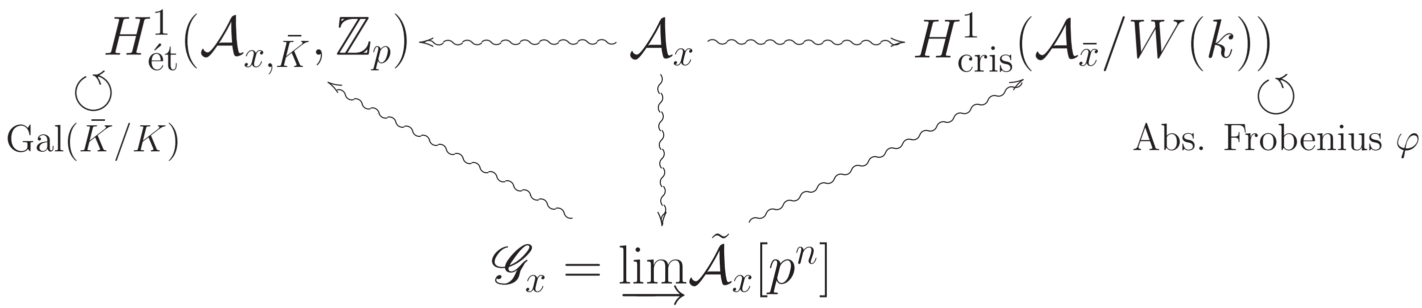
This package is the p-adic analogue of a Hodge structure of weight 1.



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To equip our integral p-adic Hodge structure with a "G-structure" and consider its deformations one needs the following

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\end{array}
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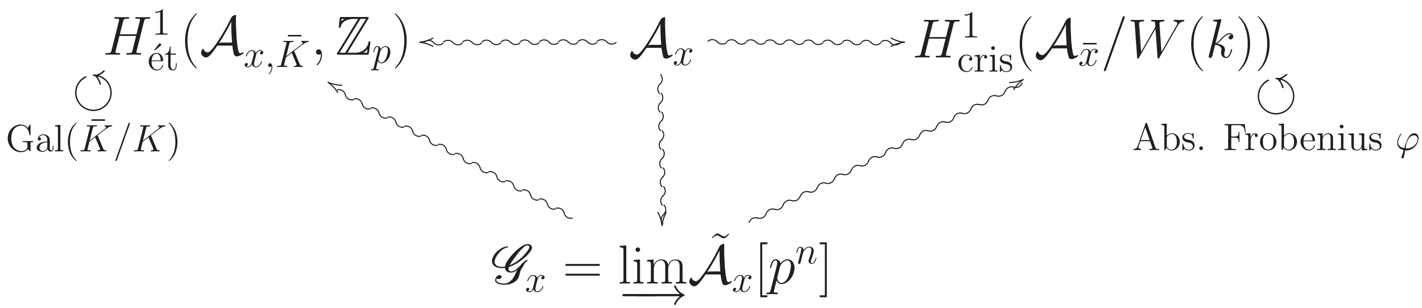

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Then using 1), one can identify this deformation space with the completion of $\mathscr{S}_{\mathsf{K}}(G, X)$ at $\bar{x}$.

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We denote by $\mathrm{Rep}^{\text{cris}}_{\mathbb{Z}_p}$ the category of crystalline $\mathbb{Z}_p$-representations.

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Let $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ denote the category of finite free $\mathfrak{S}$-modules $\mathfrak{M}$ equipped with a Frobenius semi-linear isomorphism

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The functor $\mathfrak{M}$ is still somewhat mysterious. It is constructed using p-adic Hodge theory. But see Scholze's final talk (?).

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So we have to show that P is flat over $\mathfrak{S}$ with non-empty fibres.

Claim: $P = \underline{\mathrm{Hom}}_{\mathfrak{S}, s_\alpha}(\mathfrak{M}', \mathfrak{M}) \subset \underline{\mathrm{Hom}}_{\mathfrak{S}}(\mathfrak{M}', \mathfrak{M})$ is a $G_{\mathbb{Z}_p}$-torsor.

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For R a $\mathfrak{S}$-algebra, we set $P_R = P \times_{\mathrm{Spec}\,\mathfrak{S}} \mathrm{Spec}\, R$.

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U contains the generic point by Step 1, so $\mathrm{Spec}\,\mathfrak{S}[1/p] \backslash U$ is finite.

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$$\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathbb{Z}_p} L \xrightarrow{\sim} \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathfrak{S}} \mathfrak{M},$$

$P_{\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}}$ is a trivial $G_{\mathbb{Z}_p}$-torsor.

Since $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ is faithfully flat over $\widehat{\mathfrak{S}}_{(p)}$ and $\mathfrak{S}_{(p)}$, $P_{\mathfrak{S}_{(p)}}$ is a $G_{\mathbb{Z}_p}$-torsor.

Step 2: $P_{\mathfrak{S}[1/pu]}$ is a $G_{\mathbb{Z}_p}$-torsor:

Let $U \subset \mathrm{Spec}\,\mathfrak{S}[1/up]$ the maximal open subset over which P is flat with non-empty fibres.

U contains the generic point by Step 1, so $\mathrm{Spec}\,\mathfrak{S}[1/p] \backslash U$ is finite.

Since the $\tilde{s}_\alpha$ are Frobenius invariant, and $\varphi^*(\mathfrak{M})[1/E(u)] \xrightarrow{\sim} \mathfrak{M}[1/E(u)]$ we have

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One of these conditions always hold, which contradicts finiteness.

Step 3: P_{K_0} is a $G_{\mathbb{Z}_p}$-torsor, where $\mathfrak{S} \longrightarrow K_0 := W[1/p]$ via $u \mapsto 0$.

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Since any vector bundle over U has a canonical extension to $\mathfrak{S}$, this implies that P is the trivial $G_{\mathbb{Z}_p}$-torsor.

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Example: Suppose $G_{\mathbb{Z}_p}$ is a reductive model of G, as before, let $B \subset G_{\mathbb{Z}_p} \otimes \mathbb{F}_p$ be a Borel subgroup, and take

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Theorem. *(MK-Pappas) Suppose $p > 2$, and $G_{\mathbb{Q}_p}$ splits over a tamely ramified extension. If $\lambda|p$ is a prime of E, there exists an $\mathcal{O}_{E_\lambda}$-scheme $\mathscr{S}_{\mathsf{K}_p}(G, X)$ extending $\mathrm{Sh}_{\mathsf{K}_p}(G, X)$ such that*

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This was conjectured by Rapoport and Pappas, and was known previously in some PEL cases.