Integral models of Shimura varieties I

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To $(h, g) \in X \times \mathrm{GL}_2(\mathbb{A}_f)$ we attach E_h considered up to isogeny and

$$\epsilon_{h,g} : \widehat{V}(E_h) \xrightarrow{\sim} \mathbb{A}_f^2 \xrightarrow{g} \mathbb{A}_f^2 \quad \mathrm{mod}\, \mathsf{K}.$$

The pair $(E_h, \epsilon_{h,g})$ depends only on the image of (h, g) in $\mathbb{X}_\mathsf{K}$.

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If $V_{\mathbb{Z}} \subset V$ is a $\mathbb{Z}$-lattice, and $h \in S^{\pm}$, then $V^{-1,0}/V_{\mathbb{Z}}$ is a (polarized) abelian variety, which leads to an interpretation of

$$\mathrm{Sh}_{\mathsf{K}}(\mathrm{GSp}, S^{\pm}) = \mathrm{GSp}(\mathbb{Q})\backslash S^{\pm} \times \mathrm{GSp}(\mathbb{A}_f)/\mathsf{K}$$

as a moduli space for polarized abelian varieties with level structure.

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For us the consequences will be more important than the axioms.

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$E(G, X)$ is the field of definition of this conjugacy class.

We will again denote by $\mathrm{Sh}_{\mathsf{K}}(G, X)$ this algebraic variety over $E(G, X)$.

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The condition that $\mathbb{C}^\times$ acts on $\mathrm{Lie}\, G_{\mathbb{C}}$ via

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implies Griffiths transversality.

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If these extra structures can be taken to be *endomorphisms* of the abelian variety, then (G, X) is called of PEL type.

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To do this we need integral models

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Hyperspecial K_p subgroups exist if and only if G is quasi-split at p and split over an unramified extension. This implies K_p is maximal compact. Usually one can only expect smooth models in this case.

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The $\mathscr{S}_{\mathsf{K}}(\mathrm{GSp}, S^{\pm})$ are *smooth* over $\mathcal{O}_{(\lambda)}$ if and only if the degree of the polarization in the moduli problem is prime to p. This corresponds to the condition that ψ induces a perfect pairing on $V_{\mathbb{Z}_p}$.

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One can show the same works for any R regular, formally smooth over $\mathcal{O}_{E_\lambda}$.

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The idea for such a construction goes back to Milne, Faltings, Vasiu ... The difficulty is to show that the resulting scheme is smooth.

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Many PEL cases due to Kottwitz.

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$$\mathrm{Sh}_{\mathsf{K}_p}(G, X) \hookrightarrow \mathrm{Sh}_{\mathsf{K}'_p}(\mathrm{GSp}, S^{\pm}) \hookrightarrow \mathscr{S}_{\mathsf{K}'_p}(\mathrm{GSp}, S^{\pm})$$

into a suitable moduli space of polarized abelian varieties.

The idea for such a construction goes back to Milne, Faltings, Vasiu ... The difficulty is to show that the resulting scheme is smooth.

This uses deformation theory of p-divisible groups, which can be viewed as p-adic analogues of Hodge structures of weight 1.

Theorem. *If* $p > 2$, K_p *hyperspecial and* (G, X) *is of abelian type, then* $\mathrm{Sh}_{\mathsf{K}_p}(G, X)$ *admits a smooth integral model*

$$\mathscr{S}_{\mathsf{K}_p}(G, X) = \lim_{\leftarrow\, \mathsf{K}^p} \mathscr{S}_{\mathsf{K}_p \mathsf{K}^p}(G, X)$$

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Theorem. *(MK-Pappas) Good (not smooth in general) models exist when* K_p *is parahoric,* $G_{\mathbb{Q}_p}$ *splits over a tamely ramified extension of* $\mathbb{Q}_p$*, and* $p > 2$.

Mod p points (Talk III): Suppose (G, X) is of Hodge type. We can always make a model of $\mathrm{Sh}_{\mathsf{K}}(G, X)$ by taking the closure in a moduli space for abelian varieties, as before.

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A point of $(h, g) \in \mathrm{Sh}_{\mathsf{K}_p}(G, X)(\mathbb{C})$ is called *special* if $h : \mathbb{C}^\times \to G(\mathbb{R})$ factors through $T(\mathbb{R})$, where $T \subset G$ is a subtorus, *defined over* $\mathbb{Q}$. These correspond to CM abelian varieties.

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Theorem. *(MK, Madapusi, Shin) Suppose $G_{\mathbb{Q}_p}$ is quasi-split. Then any $x \in \mathscr{S}_{\mathsf{K}}(G, X)(\bar{\mathbb{F}}_p)$ is isogenous to the reduction of a special point.*

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One has the following result, conjectured by Langlands-Rapoport in greater generality.

Theorem. *Suppose $p > 2$, K_p is hyperspecial, and (G, X) of abelian type. There exists a bijection*

$$\mathscr{S}_{\mathsf{K}_p}(G, X)(\bar{\mathbb{F}}_p) \xrightarrow{\sim} \coprod_{\varphi} I_\varphi(\mathbb{Q}) \backslash X_p(\varphi) \times X^p(\varphi)$$

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$$I_\varphi(\mathbb{Q}) \longleftrightarrow \text{automorphisms of the AV up to isogeny +extra structure}$$

and I_φ is a compact (mod center) form of the centralizer G_{γ_0}.
where $\gamma_0 \in G(\mathbb{Q})$ corresponds to the conjugacy class of Frobenius.