On the boundedness of varieties of general type

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July, 2015

Outline of the talk

Introduction

- Last time we sketched the proof of the following result for **canonical models**.

Theorem (Hacon-M^{c}Kernan-Xu)

Fix $n \in \mathbb{N}$, $C > 0$ and $\mathcal{C} \subset [0,1] \cap \mathbb{Q}$ a DCC set, then there exists an integer $r \in \mathbb{N}$ such that if (X, B) is a n-dimensional (S)LC model with $B \in \mathcal{C}$ and $(K_X + B)^n = C$, then $r(K_X + B)$ is very ample.

- Today we will see how to adapt the proof of the canonical case to the case of log canonical pairs.
- The key steps of the proof we discussed are:

Review

We first proved birational boundedness of canonical models

Theorem (Tsuji, Hacon-M^{c}Kernan, Takayama)

Fix $n \in \mathbb{N}$, then there exists $m \in \mathbb{N}$ such that if X is a canonical model, $\dim X = n$, then $|mK_X|$ is birational.

- So canonical models with $K_X^n \leq V$ are birationally bounded.
- From this, using Siu's **deformation invariance of plurigenera** and the **existence of canonical models**, we deduce the full boundedness statement.

Review

By Tsuji's argument, it suffices to show:

Theorem

Fix $n \in \mathbb{N}$, then there exists $A, B, v > 0$ such that if X is a canonical model, $\dim X = n$*, then*

1. *rK_X is birational for any $r \geq A(K_X^n)^{-1/n} + B$, and*
2. *$\mathcal{V}(n) =: \{K_X^n\}$ is discrete and $K_X^n \geq v$ for any canonical model X.*

The proof, loosely based on an argument of Anhern-Siu is by induction on the dimension and relies on a clever use of **Kawamata subadjunction**.

Outline of the talk

1. Introduction / Review
2. Boundedness of log pairs
3. The ACC for LCT's

Birational boundedness of log pairs

- Not surprisingly, the case of log pairs is substantially harder. We follow the previous proof highlighting the new ingredients.
- The first step is to show that (for fixed n, $\mathcal{C}$ and v), the set $\mathcal{LCM}$ of LC models (X, B) such that $\dim X = n$, $B \in \mathcal{C}$, $(K_X + B)^n = v$ are **log birationally** bounded.
- This means that there exists a pair $(\mathcal{Z}, \mathcal{D})$ and a projective morphism $g : \mathcal{Z} \to S$ such that for any $(X, B) \in \mathcal{LCM}$, there is an $s \in S$ and a birational morphism $h : X \dashrightarrow \mathcal{Z}_s$ such that the support of the strict transform of B plus the $\mathcal{X}_s/X$ exeptional divisors are contained in $\mathrm{Supp}(\mathcal{D}_s)$.

Birational boundedness of log pairs

- To this end, it suffices to show that there is an integer $m = O(v^{-1/n})$ such that $m(K_X + B)$ is birational.
- Then X is birationally bounded (similarly to what we have seen above for canonical pairs).
- But we must also show that the pairs are log birationally bounded.

Birational boundedness of log pairs

- WLOG, we may assume that $g : \mathcal{Z} \to S$ is a smooth morphism.
- Since we are free to replace (X, B) by a higher model $(Y, \mathbf{M}_{B,Y})$, we may assume that each $h : X \to \mathcal{Z}_s$ is a morphism.
- Let $G = \sum B_i$ where B_i are the components of $\mathrm{Supp}(B)$.
- It then suffices to show that if $H = \mathcal{O}_{\mathcal{Z}}(1)$, then $h_* G \cdot H_s^{n-1} = G \cdot h^* H_s^{n-1}$ is bounded from above.

Birational boundedness of log pairs

- Since $\min(\mathcal{C})G \leq B \leq G$, it suffices to bound the quantity

$$B \cdot h^* H_s^{n-1} = (K_X + B) \cdot h^* H_s^{n-1} - K_X \cdot h^* H_s^{n-1}.$$

- This follows as $\mathcal{Z} \to S$ is a bounded family, $K_X \cdot h^* H_s^{n-1} = K_{\mathcal{Z}_s} \cdot H_s^{n-1}$ belongs to a finite set and $(K_X + B) \cdot h^* H_s^{n-1} \leq v m^{n-1}$ (where $m = O(v^{-1/n})$).

Adjunction for log pairs

- Adjunction for log pairs is also more complicated.
- We have $K_X + B$ ample, $B \in \mathcal{C}$ and $D \sim_{\mathbb{Q}} \lambda(K_X + B)$ with $\mathcal{J}(D) = \mathcal{I}_Z$ near $x \in Z \subset X$.
- Since Z may not be of general type, we would like to find (Z^ν, Θ_{Z^ν}) LC, $\Theta_{Z^\nu} \in \mathcal{C}'$ (a DCC set) and $K_{Z^\nu} + \Theta_{Z^\nu}$ is big such that $(K_X + D + B)|_{Z^\nu} \geq K_{Z^\nu} + \Theta_{Z^\nu}$.
- We have little control over λ and the coefficients of D, but since $x \in X$ is general, we can "pretend" that Z is a fiber of a morphism $X \to T$.
- In this case $K_X|_Z = K_Z$, $\Theta_Z = B|_Z$ and we can ignore D.

Adjunction for log pairs

- In practice we have to do a delicate Kawamata's subadjunction type argument.
- Let $D(\mathcal{C}) = \{a \leq 1 | a = \frac{m-1+f}{m},\ m \in \mathbb{N},\ f = \sum f_i,\ f_i \in \mathcal{C}\}$, then $D(\mathcal{C})$ is also a DCC set.
- Let $Z^\nu \to Z$ be the normalization and $Z' \to Z^\nu$ a resolution.

Adjunction for log pairs

Theorem

There exists a divisor Θ on Z^ν, such that

1. $\Theta \in \{1-t | t \in LCT_{n-1}(D(\mathcal{C})) \cup 1\}$
2. $(K_X + D + B)|_{Z^\nu} - (K_{Z^\nu} + \Theta)$ *is PSEF, and*
3. *If Z is a general member of a covering family, then* $K_{Z'} + \mathbf{M}_{\Theta, Z'} \geq (K_X + B)|_{Z'}$ *(which is big).*

Assume that for all $d < n$ the sets

$$LCT_d(D(\mathcal{C})) = \{LCT(X, B; M) | \dim X = d, \; B \in D(\mathcal{C}), \; M \in \mathbb{N}\}$$

satisfiy the ACC property (aka the ACC for LCT's). It then follows that:

Adjunction for log pairs

- Since $K_{Z'} + \mathbf{M}_{\Theta,Z'}$ is LC and big, and the coefficients of $\mathbf{M}_{\Theta,Z'}$ are in the DCC set $\mathcal{C}' = 1 - LCT_d(D(\mathcal{C}))$ (where $d = \dim Z < n$), then by induction on the dimension

$$\mathrm{vol}(K_{Z'} + \mathbf{M}_{\Theta,Z'}) \geq v = v(d, \mathcal{C}') > 0.$$

- Thus $\mathrm{vol}(K_X + B + D)|_{Z^\nu} \geq \mathrm{vol}(K_{Z^\nu} + \Theta) \geq v$ and we can conclude similarly to the case of canonical models.

Adjunction for log pairs

- To define Θ we proceed as follows.
- After perturbing D, we may assume that on a neighborhood of the general point of Z, $(X, B + D)$ is log canonical with a unique NKLT place S above Z.

Definition of Θ

- Using the MMP, we may pick a DLT model $f : Y \to X$, that extracts only NKLT places of $(X, B + D)$ including S and is $\mathbb{Q}$-factorial. Write $\Gamma = f_*^{-1}B + \mathrm{Ex}(Y/X) - S$ and

$$K_Y+S+\Gamma+\Gamma' = g^*(K_X+B+D),\ K_S+\Phi' = (K_X+S+\Gamma+\Gamma')|_S$$

$$K_Y + S + \Gamma = f^*(K_X + B) + E, \qquad K_S + \Phi = (K_X + S + \Gamma)|_S$$

- In particular $\Gamma \in \mathcal{C}$ and $\Phi \in D(\mathcal{C})$.
- For any codimension 1 point $P \in Z^\nu$, let $t_P = LCT(S, \Phi; f^*P)$ (over the generic point of P).
- Then $\Theta = \sum(1 - t_p)P$. Define Θ' similarly for (S, Φ').
- By Kawamata subadjunction $(K_X + B + D)|_{Z^\nu} - (K_{Z^\nu} + \Theta')$ is PSEF. Since $\Theta \leq \Theta'$ we are done (with the first two claims of the theorem; the last is harder and we skip it).

Good minimal models of LC families

- A second difficulty comes from the fact that once we have a bounded family $(\mathcal{Z}, \mathcal{D}) \to S$ such that all $(X, B) \in SLC(c, n, \mathcal{C})$ are birational to a fiber $(\mathcal{Z}_s, \mathcal{D}_s)$,in order to deduce boundedness, we must take $(\mathcal{X}, \mathcal{B})$ the relative log canonical model (of a resolution) of $(\mathcal{Z}, \mathcal{D})$.
- This would require the LC mmp (and hence abundance!).
- Luckily, we can assume that our families are smooth and a dense set of fibers has a good minimal model. We show:

Good minimal models of LC families

Theorem (Hacon-M^{c}Kernan-Xu)

If $(\mathcal{Z}, \mathcal{D}) \to S$ is LC and log smooth over S and there is a point $s \in S$ such that the fiber $(\mathcal{Z}_s, \mathcal{D}_s)$ has a good minimal model, then $(\mathcal{Z}, \mathcal{D})$ has a good minimal model over S.

Deformation invariance of log plurigenera

- The key ingredient are results of Siu and Berndtson-Păun on the deformation invariance of log-plurigenera for a KLT pair and a log smooth morphism $(\tilde{\mathcal{X}}, \tilde{\mathcal{B}}) \to S$.
- In particular this implies that the generic fiber has finitely generated LC ring $R(K_{\tilde{\mathcal{X}}_\eta} + \tilde{\mathcal{B}}_\eta)$.
- So far, the only known proof of this result is analytic.

From birational boundedness to boundedness

- From this point on we may assume that our LC models (X, B) ($\dim X = n$, $B \in \mathcal{C}$, $(K_X + B)^n \leq V$) belong to a birationally bounded family.
- Recall that this means that there is a projective morphism of varieties of finite type $\mathcal{Z} \to S$ and a divisor $\mathcal{D}$ on $\mathcal{Z}$ such that for any (X, B) as above, there is a point $s \in S$ and a birational map $f : X \dashrightarrow \mathcal{Z}_s$ such that $\mathcal{D}_s$ contains the strict transform of B and the $\mathcal{Z}_s/X$ exceptional divisors.

From birational boundedness to boundedness

- Blowing up $\mathcal{Z}$ and replacing $\mathcal{D}$ by its strict transform and the exceptional divisors, we may assume that each fiber $(\mathcal{Z}_s, \mathcal{D}_s)$ is SNC.
- Replacing each (X, B) by an appropriate birational model, we may assume that each (X, B) is snc and $f : X \to \mathcal{Z}_s$ is a morphism (but $K_X + B$ is not ample; $\mathrm{vol}(K_X + B) \leq C$).

From birational boundedness to boundedness

- We begin by considering the set of all LC SNC pairs (X, B) with $B \in \mathcal{C}$ admitting a morphism to a **fixed** SNC pair $(Z, D) = (\mathcal{Z}_s, \mathcal{D}_s)$ say $f : X \to Z$ such that $f_* B \leq D$.
- **Claim:** The set $\mathcal{V} = \{\mathrm{vol}(K_X + B)\}$ satisfies the DCC.
- Throughout the proof, we are allowed to replace (X, B) by a birational pair (X', B') such that $R(K_X + B) \cong R(K_{X'} + B')$.

From birational boundedness to boundedness

- Suppose that we have a sequence of pairs (X_i, B_i) with $\mathrm{vol}(K_{X_i} + B_i) \geq \mathrm{vol}(K_{X_{i+1}} + B_{i+1})$.
- Define the b-divisor $\mathbf{D} = \lim \mathbf{M}_{B_i}$ as follows.
- Since the coefficients of $\mathbf{M}_{B_i}$ are in the DCC set $\mathcal{C}$, after passing to a subsequence, each $\lim \mathbf{M}_{B_i}(\nu)$ is well defined for any divisorial valuation ν.
- Let $\Phi = \mathbf{D}_Z$.
- Suppose that (Z, Φ) is terminal, then we claim that $R(K_{X_i} + B_i) \cong R(K_Z + f_{i,*}B_i)$ for all $i \gg 0$.

From birational boundedness to boundedness

- In fact since $f_{i,*}B_i \leq \Phi$ has finitely many components which belong to a DCC set, we may assume that for $i \gg 0$ we have $f_{i,*}B_i \leq \lim f_{i,*}B_i = \Phi$ so $(Z, f_{i,*}B_i)$ is terminal.
- But then $K_{X_i} + B_i = f_i^*(K_Z + f_{i,*}B_i) + E_i$ with $E_i \geq 0$ and f_i-exceptional.
- Thus $H^0(m(K_{X_i} + B_i)) = H^0(m(K_Z + f_{i,*}B_i))$ for all $m > 0$.
- Thus we may assume that $X_i = Z$ for all $i \gg 0$.
- Suppose that $\mathrm{vol}(K_Z + B_i) \geq \mathrm{vol}(K_Z + B_{i+1})$. Passing to a subsequence, we may assume $B_i \leq B_{i+1}$, so that $\mathrm{vol}(K_Z + B_i) \leq \mathrm{vol}(K_Z + B_{i+1})$.
- Thus $\mathrm{vol}(K_{X_i} + B_i) = \mathrm{vol}(K_{X_{i+1}} + B_{i+1})$ for all $i \gg 0$.

From birational boundedness to boundedness

- The statement about finiteness of log canonical models is related to a general result of the MMP.

Theorem (Birkar-Cascini-Hacon-M^{c}Kernan)

Let X be a smooth variety and $B_1 \leq B_2$ effective divisors with SNC such that $K_X + B_1$ is big and $K_X + B_2$ is KLT. Then there is a finite set of birational maps $(\psi_i : X \dashrightarrow W_i)_{i \in I}$ such that for any $\mathbb{Q}$-divisor $B_1 \leq B \leq B_2$, there exists an index $i \in I$ such that ψ_i is the LCM of (X, B) and in particular $\mathrm{Proj}(R(K_X + B)) \cong W_i$.

- Next we explain how to deal with the case when (Z, Φ) is not terminal.

From birational boundedness to boundedness

- Suppose that (Z, Φ) is KLT. Then it is easy to see that blowing up Z finitely many times along strata of Φ (and the exceptional divisors), we obtain a birational morphism $h : Z' \to Z$ such that $K_{Z'} + \Phi' = h^*(K_Z + \Phi)$, $\Phi' \geq 0$, and (Z', Φ') is terminal.
- The hardest case is when (Z, Φ) is log canonical but not KLT. The proof proceeds by induction on the codimension of the smallest LC center.

From birational boundedness to boundedness

- If $\mathbf{D} \geq \mathbf{L}_{\Phi}$, then we find a contradiction to $\mathrm{vol}(K_{X_i} + B_i) > \mathrm{vol}(K_{X_{i+1}} + B_{i+1})$.
- Note that then $\mathrm{vol}(K_Z + \Phi) > \mathrm{vol}(K_{X_i} + B_i)$. However $\mathrm{vol}(K_Z + \Phi) = \lim \mathrm{vol}(K_Z + (1-\epsilon)\Phi)$ and so the contradiction follows if we show $\lim \mathrm{vol}(K_{X_i} + B_i) \geq \mathrm{vol}(K_Z + (1-\epsilon)\Phi)$.
- But $(Z, (1-\epsilon)\Phi)$ is KLT and we can use the terminalization trick explained above to get $\mathbf{M}_{B_i, Z'} \geq \mathbf{L}_{(1-\epsilon)\Phi, Z'}$ and hence the required inequality.

From birational boundedness to boundedness

- So assume that there is a divisor with valuation ν over Z such that $\mathbf{D}(\nu) < \mathbf{L}_{Z,\Phi}$. In particular $\mathbf{L}_{Z,\Phi} > 0$ and so ν is a toric valuation.
- Let $\mu : Z_\nu \to Z$ be the corresponding toric blow up. Set $\Phi_\nu = \mu_*^{-1}\Phi + d_\nu E_\nu$ where E_ν is the exceptional divisor and $0 \leq d_\nu = \mathbf{D}(\nu) < 1$.
- We may replace (Z, Φ) by (Z_ν, Φ_ν) and (X_i, B_i) by $X_{i,\nu} \to X_i$ (extracting the divisor corresponding to ν if necessary) and B_i by the strict transform of B_i and the exceptional divisor $E_{i,\nu}$ corresponding to ν with $L_{B_i}(\nu)$.
- Then the only remaining NKLT centers have codimension $\geq n - 1$.....

Boundedness in families

- We must now show that the analogous statements hold when (X, B) is birational to a fiber of a family $(\mathcal{Z}, \mathcal{D}) \to S$.
- Decomposing S in to a finite disjoint union of locally closed subsets (and applying base change), we can assume that each strata of $(\mathcal{Z}, \mathcal{D})$ is smooth with connected fibers over S.

Boundedness in families

- By a result of Siu, Hacon-M^cKernan, Berndtson-Păun, Hacon-M^ckernan-Xu, the log plurigenera $h^0(m(K_{\mathcal{Z}_s}+\mathcal{B}_s))$ are deformation invariant for any divisor $0 \leq \mathcal{B} \leq \mathcal{D}$.
- Suppose again for simplicity that $(\mathcal{Z}, \mathcal{B})$ is terminal, then for any (X, B) we have $h^0(m(K_X+B)) = h^0(m(K_{\mathcal{Z}_s}+\mathcal{B}_s))$ and so the set of volumes $V = \{\mathrm{vol}(K_X+B)\}$ is determined by the volumes of finitely many fibers $(\mathcal{Z}_s, \mathcal{B}_s)$ (one for each component of s).

Outline of the talk

ACC for LCT's

Theorem (Hacon-M[c]Kernan-Xu)

Fix $n \in \mathbb{N}$ and $\mathcal{C} \subset [0.1]$ a DCC set. Let $LCT_n(\mathcal{C}) = \{LCT(X, B; M)\}$ where (X, B) is LC, $B \in \mathcal{C}$, $M \geq 0$ is a $\mathbb{Z}$-Weil, $\mathbb{Q}$-Cartier divisor. Then $LCT_n(\mathcal{C})$ satisfies the DCC.

- This is Shokurov's ACC for LCT's conjecture, which was proved in the case that X has bounded singularities by Ein-Mustață-de Fernex.
- We will now give a sketch of the proof.

ACC for LCT's

- Suppose that there is a sequence of pairs (X_i, B_i) and divisors M_i as above with $t_i = LCT(X_i, B_i; M_i)$ such that $t_i < t_{i+1}$ for all $i > 0$.
- We let $t = \lim t_i > t_i$.
- For all i, let $\nu_i : Y_i \to X_i$ be a proper birational morphism extracting a unique divisor of discrepancy -1 with center a minimal NKLT center of $(X_i, B_i + t_i M_i)$.
- By induction on the dimension, we may assume that this minimal NKLT center is a point $x_i \in X_i$.
- We may assume that $\rho(Y_i/X_i) = 1$.

ACC for LCT's

- Define $K_{E_i} + \Delta_i = (K_{Y_i} + E_i + \nu_{i,*}^{-1}(B_i + t_i M_i))|_{E_i} \equiv 0$, and $K_{E_i} + \Delta'_i = (K_{Y_i} + E_i + \nu_{i,*}^{-1}(B_i + tM_i))|_{E_i}$.
- Note that the coefficients of $B_i + t_i M_i$ and $B_i + tM_i$ are in the DCC set $\mathcal{C}' = \mathcal{C} \cup \{t_i | i \in \mathbb{N}\} \cup \{t\mathbb{N}\}$ and hence the coefficients of Δ_i and Δ'_i are in the DCC set $D(\mathcal{C}')$.
- Since $t > t_i$ and $(\nu_{i,*}^{-1} M_i)|_{E_i} \neq 0$, then $K_{E_i} + \Delta'_i$ is ample.
- Since $\lim t_i = t$, $K_{E_i} + \Delta'_i$ is LC by the ACC for LCT's in dimension $n-1$.

ACC for LCT's

- The following consequence of the results on the boundedness of LC models gives an immediate contradiction:
- **Claim:** There exists a number $\tau < 1$ such that for all i, $K_{E_i} + \tau\Delta'_i$ is big.
- But then since we may assume $\tau\Delta'_i < \Delta_i$ for $i \gg 0$, it follows that

$$0 < \mathrm{vol}(K_{E_i} + \tau\Delta'_i) \leq \mathrm{vol}(K_{E_i} + \Delta_i) = 0$$

  which is impossible.
- We now verify the claim.

ACC for LCT's

- The idea is that there is an integer m (depending only on the dimension n and the DCC set $\mathcal{C}$) such that if (X, B) is a LC model with $K_X + B$ ample, then $m(K_X + B)$ is birational (even for $\mathbb{R}$-divisors).
- But then, $h^0(K_X + (mn+1)(K_X + B)) > 0$.
- Since $K_X + (mn+1)(K_X + B) = (mn+2)(K_X + \alpha B)$, where $\alpha = (mn+1)/(mn+2) < 1$, we let $\tau = (\alpha + 1)/2$.

ACC for LCT's

- To see this ($h^0(K_X + (mn+1)(K_X + B)) > 0$), pick a general point $x \in X$ and a divisor $D \sim_{\mathbb{Q}} \frac{n}{n+1}(H_1 + \ldots + H_{n+1})$ where the H_i correspond to general hyperplanes through x.
- It is easy to see that $\mathcal{J}(D) = \mathfrak{m}_x$ near $x \in X$.
- By Nadel vanishing $H^1(K_X + (mn+1)(K_X + B) \otimes \mathcal{J}(D)) = 0$ and hence $K_X + (mn+1)(K_X + B)$ is generated at $x \in X$.