On the boundedness of varieties of general type

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Outline of the talk

1. Introduction and Preliminaries
2. Boundedness of Canonical models
The goal of the remaining two lectures is to sketch the proof of the following.

**Theorem (Hacon-McKernan-Xu)**

Fix \( n \in \mathbb{N}, \ C > 0 \) and \( C \subset [0, 1] \cap \mathbb{Q} \) a DCC set, then there exists an integer \( r \in \mathbb{N} \) such that if \((X, B)\) is a \( n \)-dimensional SLC model with \( B \in C \) and \((K_X + B)^n = C\), then \( r(K_X + B) \) is very ample.

- SLC model: \((X, B)\) projective SLC pair over \( \mathbb{C} \), \( K_X + B \) ample (we will focus on LC models).
- Alexeev \( n = 2 \).
- We begin by reviewing a few preliminaries.
Let $D$ be an $\mathbb{R}$-divisor on a normal proper $n$-dimensional variety $X$ and define the **volume** of $D$ by

$$\text{vol}(D) = \lim_{m \to \infty} \frac{h^0(mD)}{m^n/n!} = \lim_{m \to \infty} \frac{h^0(mD)}{m^n/n!}.$$ 

We say that $D$ is **big** iff $\text{vol}(D) > 0$ in which case, by a lemma of Fujita, we may write $D \sim_\mathbb{Q} A + E$ where $A$ is an ample $\mathbb{Q}$-Cartier divisor and $E \geq 0$. 

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If $D$ is ($\mathbb{Q}$-Cartier and) nef, so that $D \cdot C \geq 0$ for any curve $C \subset X$, then $\text{vol}(D) = D^n$.

It is easy to see that if $f : X \to Y$ is a birational morphism of normal projective varieties, then $\text{vol}(f_* D) \geq \text{vol}(D)$ and

if $G$ is an $\mathbb{R}$-Cartier divisor on $Y$ such that $D - f^* G$ is effective and $f$-exceptional, then $\text{vol}(D) = \text{vol}(G)$.
We will often apply this to log pairs in the following form.

Let \((X, B)\) be a log pair \((K_X + B \text{ is } \mathbb{R}-\text{Cartier})\) and define the following \(b\)-divisors. If \(f : Y \to X\) is a resolution and \(f_* K_Y = K_X\), then

\[
K_Y = f^*(K_X+B)+A_{B,Y}, \quad K_Y+L_{B,Y} = f^*(K_X+B)+E_{B,Y}
\]

where \(L_{B,Y}, E_{B,Y} \geq 0\) and \(L_{B,Y} \land E_{B,Y} = 0\) (for \(a, b \in \mathbb{R}\), \(a \land b = \min\{a, b\}\)).
Volumes

- We also let $M_{B,Y} = f^{-1}_* B + \text{Ex}(f)$ so that $M_{B,Y} \geq L_{B,Y}$ if $(X, B)$ is log canonical and then

$$\text{vol}(K_X + B) = \text{vol}(K_Y + L_{B,Y}) = \text{vol}(K_Y + M_{B,Y}).$$

- If $G$ is an $\mathbb{R}$-Cartier divisor on $Y$ with $f_* G = B$, then we also have $\text{vol}(K_Y + G) = \text{vol}(K_Y + L_{B,Y} \wedge G)$. Since

$$f_* H^0(Y, m(K_Y + G)) \subset H^0(X, m(K_X + B)) \cong H^0(Y, m(K_Y + L_{B,Y})).$$
We will need the following easy but important result.

**Theorem (Easy addition)**

Let $f : X \rightarrow S$ be a projective morphism of smooth varieties, then
\[
\kappa(X) \leq \kappa(X_s) + \dim S \text{ where } s \in S \text{ is general. In particular if } X \text{ has general type, then so does } X_s.
\]

• Recall that $X$ is of general type if (replacing $X$ by a resolution) $\kappa(X) = \dim X$, i.e. if $\text{vol}(K_X) > 0$.
• By definition $\kappa(X) = \text{tr.deg.}_\mathbb{C} R(K_X) - 1$. 
Thus, if $X$ is of general type then $X_s$ is of general type.

The idea is as follows. If $X$ has general type (the other cases are similar), we may write $K_X \sim_{\mathbb{Q}} A + E$ where $A$ is ample and $E$ is effective.

But then $K_{X_s} = K_{X}|_{X_s} \sim_{\mathbb{Q}} A|_{X_s} + E|_{X_s}$ where $A|_{X_s}$ is ample and $E|_{X_s}$ is effective.

We have the following important consequence.
Theorem

Let $Z \to T$ be a projective morphism and $f : Z \to X$ a dominant morphism to a projective variety. If $X$ is of general type, then so is $Z_t$ for general $t \in T$. 

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- Cutting by generic hyperplanes on $T$, we may assume that $Z \to X$ is generically finite.
- Replacing $X$ and $Z \to T$ by appropriate birational models we may assume that $X, Z, T$ are smooth.
- Since $f : Z \to X$ is generically finite, we have $K_Z = f^*K_X + R$ where $R \geq 0$ is the ramification divisor.
- Thus $Z$ is also of general type.
- By the easy addition theorem, $Z_t$ is of general type.
Let $(X, B)$ be a **log pair** so that $X$ is normal and $B$ is an $\mathbb{R}$-divisor and $K_X + B$ is $\mathbb{R}$-Cartier.

The $b$-divisors $A, L, E$ were defined so that $A = E - L$ and

$$K_Y + L_{B,Y} = f^*(K_X + B) + E_{B,Y}, \quad L_{B,Y} \wedge E_{B,Y} = 0.$$  

We say that $(X, B)$ is **log canonical / LC** resp. **Kawamata log terminal / KLT** if for any prime divisor $E$ over $X$, we have $\text{mult}_E(L) \leq 1$, resp. $\text{mult}_E(L) < 1$ (i.e. $\text{mult}_E(A) > -1$).
We say that \((X, B)\) is **canonical** (resp. **terminal**) if 
\[ \text{mult}_E(A_B) \geq 0 \] (resp. \[ \text{mult}_E(A_B) > 0 \]) for all divisors \(E\)
exceptional over \(X\) (in particular \(L_{B,Y} = f_*^{-1}B\)).

We say that \((X, S + B)\) is **PLT** if \(S = \lfloor S + B \rfloor\) is a prime divisor and 
\[ \text{mult}_E(L_{B,Y}) < 1 \] for any other prime divisor over \(X\).
The coefficients of $A = L - E$ are a measure of the singularities of $(X, B)$. The infimum of these coefficients is the **total discrepancy**.

More negative total discrepancies (resp. discrepancies) correspond to more singular pairs. $(\geq -1, > -1$, (resp. $\geq 0$, $> 0$) correspond to LC, KLT, (resp. canonical, terminal)).
The log canonical and KLT conditions can be checked on one (any) log resolution.

If $f: Y \to X$ is the blow up of the vertex of a cone over a rational curve of degree $n$ with exceptional curve $E$, then by adjunction

$$-2 = (K_Y + E) \cdot E = (\nu^*(K_X) + (a_E + 1)E) \cdot E = -n(a_E + 1).$$

(Since $E^2 = -n$.) Thus $a_E = -1 + \frac{2}{n}$.

The same computation shows that if the curve is elliptic, then $a_E = -1$ and if the curve has genus $g \geq 2$, then $a_E < -1$.

Note that if $a_E < -1$ then the discrepancy is $-\infty$. 
Singularities of the MMP

- KLT singularities are rational ($R^i f_* O_Y = 0$ for $i > 0$) and LC singularities are Du Bois (Kollár-Kovács).
- Terminal/canonical singularities arise from minimal/canonical models of smooth varieties $X$ and LC/KLT singularities are the singularities of log canonical models $\text{Proj}(R(K_X + B))$ where $(X, B)$ is a SNC pair with coefficients $0 \leq b_i \leq 1/0 \leq b_i < 1$. 
If \( a_P := \text{mult}_P A_B \leq -1 \) (resp. \( < -1 \)), then we say that \( P \) is a **NKLT place** (resp. a **NLC place**) and its image \( f(P) \) is a **center** of NKLT (resp. NLC) singularities.

If \((X, B)\) is LC and \( G \geq 0 \) is an \( \mathbb{R} \)-Cartier divisor, then the **log canonical threshold** is

\[
lct(X, B; G) = \sup\{ c > 0 | (X, D + cG) \text{ is LC} \}.
\]

One can compute log canonical thresholds on a single log resolution (eg, \( \text{lct}(\mathbb{C}^2, 0; \{ y^2 = x^3 \}) = 5/6 \)).
Multiplier ideals (5 minute refresher)

- Let $X$ be smooth, $B \geq 0$, $f : Y \to X$ a log resolution of $(X, B)$ then the **multiplier ideal sheaf** of $(X, D)$ is

  \[ J = J(X, B) = f_* \mathcal{O}_Y(K_{Y/X} - [f^*B]) \subset f_* \mathcal{O}_Y(K_{Y/X}) = \mathcal{O}_X. \]

- $J$ is independent of the log resolution.
- $J = \mathcal{O}_X$ iff $(X, B)$ is KLT (as $K_{Y/X} - [f^*B] = [A_Y]$).
- If $B$ is SNC, then $J(B) = \mathcal{O}_X(-[B])$.
- If $G$ Cartier, then $J(G + B) = J(B) \otimes \mathcal{O}_X(-G)$.
- $D_1 \leq D_2$ then $J(D_2) \subset J(D_1)$. 
Multiplier ideals

Multiplier ideals $\mathcal{J}(D)$ are a sophisticated measure of the singularities of $D$.

- $\text{mult}_x(D) \geq n = \dim X$, then $\mathcal{J}(D) \subset \mathfrak{m}_x$ (just blow up $x \in X$).
- (Harder) If $\text{mult}_x(D) < 1$, then $\mathcal{J}(D)_x = \mathcal{O}_{X,x}$.
- $\text{lct}(X, 0; G) = \sup\{t | \mathcal{J}(X, tG) = \mathcal{O}_X\}$. 
An easy consequence of Kawamata-Viehweg’s vanishing theorem is the following.

**Theorem (Nadel vanishing)**

Let $X$ be smooth, $f : X \to Z$ a projective morphism, $D \geq 0$ an $\mathbb{R}$-divisor, $N$ a Cartier divisor such that $N - D$ is $f$-nef and $f$-big, then

$$R^i f_* (\mathcal{O}_X (K_X + N) \otimes \mathcal{J}(D)) = 0 \quad \forall i > 0.$$  

At first sight $\mathcal{J}(D)$ is a technical annoyance, but as we will see later it should be viewed as an opportunity.
If $H$ is a smooth divisor on a smooth variety $X$, $D \geq 0$ an effective $\mathbb{R}$-divisor on $X$ whose support does not contain $H$. Then $\mathcal{J}(H, D|_H) \subset \mathcal{J}(X, D) \cdot \mathcal{O}_H$, and

if $0 < s < 1$, then for all $0 < t \ll 1$ we have

$$\mathcal{J}(X, D + (1 - t)H) \cdot \mathcal{O}_H \subset \mathcal{J}(H, (1 - s)D|_H).$$

This is an example of inversion of adjunction. If $\mathcal{J}(H, (1 - s)D|_H) \subset \mathfrak{m}_x$ ($x \in H$ and $0 < s < 1$), then $\mathcal{J}(X, D + (1 - t)H) \subset \mathfrak{m}_x$ for all $0 < t \ll 1$. 
(Analog for log pairs) $(X, S + B)$ an effective log pair, $
u : S^\nu \to S$ the normalization of $S$ and $K_{S^\nu} + B_{S^\nu} = \nu^*(K_X + S + B)$, then

1. $(X, S + B)$ is PLT iff $(S^\nu, B_{S^\nu})$ is KLT and
2. $(X, S + B)$ is LC iff $(S^\nu, B_{S^\nu})$ is LC.

The first is an easy consequence of the connectedness lemma of Kollár and Shokurov, the second is a deep result of Kawakita.
Outline of the talk

1. Introduction and Preliminaries
2. Boundedness of Canonical models
Theorem (Tsuji, Hacon-McKernan, Takayama)

Fix $n \in \mathbb{N}$ and $V > 0$, then there exists $m \in \mathbb{N}$ such that if $X$ is a canonical model, $\dim X = n$ and $K_X^n \leq V$, then $mK_X$ is very ample. (In particular $\{K_X^n\}$ is discrete.)

Tsuji’s idea is to first prove the following weaker result that $mK_X$ is birational and to do this by induction on $n = \dim X$ by first proving the following weaker statements.
Theorem

Fix $n \in \mathbb{N}$, then there exists $A, B, \nu > 0$ such that if $X$ is a canonical model, $\dim X = n$, then

1. $rK_X$ is birational for any $r \geq A(K_X^n)^{-1/n} + B$, and
2. $\mathcal{V}(n) =: \{ K_X^n \}$ is discrete and $K_X^n \geq \nu$ for any canonical model $X$.

Note that $(1)_n + (2)_n$ imply that $rK_X$ is birational for any integer $r \geq A\nu^{-1/n} + B$. 
Birational boundedness

To see that $(1)_n$ implies $(2)_n$, we may assume that $K^n_X \leq V$ e.g. $K^n_X \leq 1$ and let $m = \lceil A(K^n_X)^{-1/n} + B \rceil$.

If $Z$ is the closure of $\phi|_{mK_X}(X)$, then

$$\deg(Z) \leq (mK_X)^n < (A(K^n_X)^{-1/n} + B + 1)^n K^n_X < (A + B + 1)^n.$$ 

Therefore $Z$ belongs to a bounded family $\mathcal{Z} \rightarrow S$. 

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If $Z$ is the closure of $\phi|_{mK^n_X|(X)}$, then

$$\deg(Z) \leq (mK^n_X)^n < (A(K^n_X)^{-1/n} + B + 1)^n K^n_X < (A + B + 1)^n.$$ 

Therefore $Z$ belongs to a bounded family $\mathcal{Z} \to S$.

Let $\mathcal{X}' \to \mathcal{Z}$ be a resolution of this family.

Decomposing $S$ into a finite union of locally closed subsets, we may assume that $\mathcal{X}' \to S$ is a smooth morphism and

$$\{s \in S | \kappa(\mathcal{X}'_s) = \dim \mathcal{X}'_s \}$$

is dense.

By deformation invariance of plurigenera, all fibers $\mathcal{X}'_s$ are of general type and

$$\{\text{vol}(K\mathcal{X}'_s) | s \in S \}$$

is finite and $(2)_n$ follows.
Let $\mathcal{X} \to S$ be the relative canonical model. Then for any canonical model $X$ with $\dim X = n$ and $\text{vol}(K_X) < V$, we have $s \in S$ s.t. $X \cong \mathcal{X}_s$.

It is easy to see that there is an integer $m \in \mathbb{N}$ such that $mK_{\mathcal{X}_s}$ is very ample for all $s \in S$ and hence $mK_X$ is very ample for any canonical model of dimension $n$ with $K^n_X \leq V$. (In particular $\text{vol}(K_X) \geq 1/m^n$.)

Thus, the set of all $n$-dimensional canonical models with volume bounded from above $0 < K^n_X \leq V$ is bounded.
It remains to show that \((1)_{n-1} + (2)_{n-1}\) implies \((1)_{n}\).

The main idea is, for any very general points \(x, y \in X\), to produce a divisor \(D \sim_{\mathbb{Q}} \lambda K_X\) such that:

1. \(\mathcal{J}(D) \subset m_y\) and on a neighborhood of \(x\), \(\mathcal{J}(D) = m_x\), and
2. \(\lambda < m - 1 = \lceil A(K^n_X)^{-1/n} + B \rceil - 1\).
Birational boundedness

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  2. \(\lambda < m - 1 = \lceil A(K^n_X)^{-1/n} + B \rceil - 1\).
- By Nadel vanishing (assuming for simplicity that \(X\) is smooth), \(H^1(\omega^m_X \otimes \mathcal{J}(D)) = 0\) and hence \(H^0(\omega^m_X) \to H^0(\omega^m_X \otimes O_X/\mathcal{J}(D))\) is surjective.
Birational boundedness

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  1. $\mathcal{J}(D) \subset m_y$ and on a neighborhood of $x$, $\mathcal{J}(D) = m_x$, and
  2. $\lambda < m - 1 = \lceil A(K^n_X)^{-1/n} + B \rceil - 1$.
- By Nadel vanishing (assuming for simplicity that $X$ is smooth), $H^1(\omega^m_X \otimes \mathcal{J}(D)) = 0$ and hence $H^0(\omega^m_X) \to H^0(\omega^m_X \otimes \mathcal{O}_X/\mathcal{J}(D))$ is surjective.
- Since $\mathcal{C}_x$ is a summand of $\mathcal{O}_X/\mathcal{J}(D)$, and $\mathcal{J}(D) \subset m_y$ we have produced sections vanishing at $y$ but not at $x$. 
Birational boundedness

- It remains to show that $(1)_{n-1} + (2)_{n-1}$ implies $(1)_n$.
- The main idea is, for any very general points $x, y \in X$, to produce a divisor $D \sim_{\mathbb{Q}} \lambda K_X$ such that
  1. $\mathcal{J}(D) \subset \mathfrak{m}_y$ and on a neighborhood of $x$, $\mathcal{J}(D) = \mathfrak{m}_x$, and
  2. $\lambda < m - 1 = \lceil A(K_X^n)^{-1/n} + B \rceil - 1$.
- By Nadel vanishing (assuming for simplicity that $X$ is smooth), $H^1(\omega_X^m \otimes \mathcal{J}(D)) = 0$ and hence $H^0(\omega_X^m) \to H^0(\omega_X^m \otimes \mathcal{O}_X/\mathcal{J}(D))$ is surjective.
- Since $\mathcal{C}_x$ is a summand of $\mathcal{O}_X/\mathcal{J}(D)$, and $\mathcal{J}(D) \subset \mathfrak{m}_y$ we have produced sections vanishing at $y$ but not at $x$.
- This implies that $mK_X$ is birational.
- In what follows we will focus on the condition $\mathcal{J}(D) = \mathfrak{m}_x$ (it is not much harder to guarantee $\mathcal{J}(D) \subset \mathfrak{m}_y$.)
Birational boundedness

- We have $h^0(mK_X) = \frac{m^n}{n!} K^n_X + o(m^n)$ whilst vanishing at a smooth point to order $k$ is $k^n/n! + o(k^n)$ conditions.
- Thus we will find a divisor $D_1 \sim_{\mathbb{Q}} \lambda_1 K_X$ with $\text{mult}_X(D_1) \geq n$ and $\lambda_1 = O((K^n_X)^{-1/n})$.
- Locally we have $\mathcal{J}(D_1) = \mathcal{I}_Z \subset m_X$ where we may assume that $Z$ is reduced and irreducible.
- Our goal is to cut down $Z$ to a point.
To do this, we need to bound \((K_X|_Z)^{\dim Z}\) from below.

Since \(x \in X\) is general, \(Z\) is of general type and so by induction on the dimension \(\vol(K_{Z'}) \geq \nu(d) > 0\) where \(d = \dim Z\) and \(Z' \to Z\) is a resolution.

Tsuji’s idea is to use Kawamata’s subadjunction to compare \(\vol(K_{Z'})\) and \(\vol(K_X|_Z) = (K_X|Z)^{\dim Z}\).
Recall that $\mathcal{J}(D_1) = \mathcal{I}_Z$ near $x \in Z \subset X$ and $D_1 \sim_{\mathbb{Q}} \lambda_1 K_X$.

Assume for simplicity that $Z$ is smooth.

Then, by Kawamata subadjunction (as $K_X$ is ample)\[ (1 + \lambda_1 + \epsilon)K_X|_Z \sim_{\mathbb{Q}} (K_X + D_1 + \epsilon K_X)|_Z \geq_{\mathbb{Q}} K_Z \]

and so $(K_X|_Z)^d \geq \left(\frac{1}{1+\lambda_1}\right)^d \cdot \nu(d)$ where $d = \text{dim } Z$. 
We now pick a very general point $x' \in Z$ and $D'_Z \sim_{\mathbb{Q}} \lambda' K_X|_Z$ such that $\text{mult}_{x'}(D'_Z) > d$ and $\lambda' \leq d(1 + \lambda_1) + 1$.

Since $K_X$ is ample, by Serre vanishing we may assume that $D'_Z = D'|_Z$ where $D' \sim_{\mathbb{Q}} \lambda' K_X$.

By inversion of adjunction

$m_{x'} \supseteq \mathcal{I}((1 - \delta)D_1 + (1 - \eta)D') = \mathcal{I}_{Z_2}$ where $\text{dim } Z_2 < \text{dim } Z$. 
Let $D_2 = (1 - \delta)D_1 + (1 - \eta)D'$ so that $D_2 \sim \lambda_2 K_X$ where $
abla_2 = O((K^n_X)^{-1/n})$.

Note that $x'$ is also a very general point of $X$.

Thus we may replace $x, D_1, Z, \lambda_1$ by $x', D_2, Z_2, \lambda_2$.

After $\leq n$ iterations, we may assume that $\dim Z = 0$ i.e. that $\mathcal{J}(D) = \mathfrak{m}_x$ on a neighbourhood of $x \in X$. QED