On the boundedness of varieties of general type

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Outline of the talk

Introduction

- The goal of the remaining two lectures is to sketch the proof of the following.

Theorem (Hacon-M^{c}Kernan-Xu)

Fix $n \in \mathbb{N}$, $C > 0$ and $\mathcal{C} \subset [0,1] \cap \mathbb{Q}$ a DCC set, then there exists an integer $r \in \mathbb{N}$ such that if (X, B) is a n-dimensional SLC model with $B \in \mathcal{C}$ and $(K_X + B)^n = C$, then $r(K_X + B)$ is very ample.

- SLC model: (X, B) projective SLC pair over $\mathbb{C}$, $K_X + B$ ample (we will focus on LC models).
- Alexeev $n = 2$.
- We begin by reviewing a few preliminaries.

Volumes

- Let D be an $\mathbb{R}$-divisor on a normal proper n-dimensional variety X and define the **volume** of D

$$\mathrm{vol}(D) = \overline{\lim} \frac{h^0(mD)}{m^n/n!} = \lim \frac{h^0(mD)}{m^n/n!}.$$

- We say that D is **big** iff $\mathrm{vol}(D) > 0$ in which case, by a lemma of Fujita, we may write $D \sim_{\mathbb{Q}} A + E$ where A is an ample $\mathbb{Q}$-Cartier divisor and $E \geq 0$.

Volumes

- If D is ($\mathbb{Q}$-Cartier and) nef, so that $D \cdot C \geq 0$ for any curve $C \subset X$, then $\mathrm{vol}(D) = D^n$.
- It is easy to see that if $f : X \to Y$ is a birational morphism of normal projective varieties, then $\mathrm{vol}(f_* D) \geq \mathrm{vol}(D)$ and
- if G is an $\mathbb{R}$-Cartier divisor on Y such that $D - f^* G$ is effective and f-exceptional, then $\mathrm{vol}(D) = \mathrm{vol}(G)$.

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Volumes

- We will often apply this to log pairs in the following form.
- Let (X, B) be a log pair ($K_X + B$ is $\mathbb{R}$-Cartier) and define the following b-divisors. If $f : Y \to X$ is a resolution and $f_* K_Y = K_X$, then

$$K_Y = f^*(K_X+B)+\mathbf{A}_{B,Y}, \qquad K_Y+\mathbf{L}_{B,Y} = f^*(K_X+B)+\mathbf{E}_{B,Y}$$

where $\mathbf{L}_{B,Y}, \mathbf{E}_{B,Y} \geq 0$ and $\mathbf{L}_{B,Y} \wedge \mathbf{E}_{B,Y} = 0$ (for $a, b \in \mathbb{R}$, $a \wedge b = \min\{a, b\}$).

Volumes

- We also let $\mathbf{M}_{B,Y} = f_*^{-1}B + \mathrm{Ex}(f)$ so that $\mathbf{M}_{B,Y} \geq \mathbf{L}_{B,Y}$ if (X, B) is log canonical and then

$$\mathrm{vol}(K_X + B) = \mathrm{vol}(K_Y + \mathbf{L}_{B,Y}) = \mathrm{vol}(K_Y + \mathbf{M}_{B,Y}).$$

- If G is an $\mathbb{R}$-Cartier divisor on Y with $f_*G = B$, then we also have $\mathrm{vol}(K_Y + G) = \mathrm{vol}(K_Y + \mathbf{L}_{B,Y} \wedge G)$. Since

$$f_*H^0(Y, m(K_Y+G)) \subset H^0(X, m(K_X+B)) \cong H^0(Y, m(K_Y+\mathbf{L}_{B,Y})).$$

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Kodaira dimension

- We will need the following easy but important result.

Theorem (Easy addition)

Let $f : X \to S$ be a projective morphism of smooth varieties, then $\kappa(X) \leq \kappa(X_s) + \dim S$ where $s \in S$ is general. In particular if X has general type, then so does X_s.

- Recall that X is of general type if (replacing X by a resolution) $\kappa(X) = \dim X$, i.e. if $\mathrm{vol}(K_X) > 0$.
- By definition $\kappa(X) = \mathrm{tr.deg.}_{\mathbb{C}} R(K_X) - 1$.

Kodaira dimension

- Thus, if X is of general type then X_s is of general type.
- The idea is as follows. If X has general type (the other cases are similar), we may write $K_X \sim_{\mathbb{Q}} A + E$ where A is ample and E is effective.
- But then $K_{X_s} = K_X|_{X_s} \sim_{\mathbb{Q}} A|_{X_s} + E|_{X_s}$ where $A|_{X_s}$ is ample and $E|_{X_s}$ is effective.
- We have the following important consequence.

Kodaira dimension

Theorem

Let $Z \to T$ be a projective morphism and $f : Z \to X$ a dominant morphism to a projective variety. If X is of general type, then so is Z_t for general $t \in T$.

Kodaira dimension

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- Cutting by generic hyperplanes on T, we may assume that $Z \to X$ is generically finite.
- Replacing X and $Z \to T$ by appropriate birational models we may assume that X, Z, T are smooth.
- Since $f : Z \to X$ is generically finite, we have $K_Z = f^*K_X + R$ where $R \geq 0$ is the ramification divisor.
- Thus Z is also of general type.
- By the easy addition theorem, Z_t is of general type.

Singularities of the MMP (5 minute refresher)

- Let (X, B) be a **log pair** so that X is normal and B is an $\mathbb{R}$-divisor and $K_X + B$ is $\mathbb{R}$-Cartier.
- The b-divisors $\mathbf{A}, \mathbf{L}, \mathbf{E}$ were defined so that $\mathbf{A} = \mathbf{E} - \mathbf{L}$ and

$$K_Y + \mathbf{L}_{B,Y} = f^*(K_X + B) + \mathbf{E}_{B,Y}, \qquad \mathbf{L}_{B,Y} \wedge \mathbf{E}_{B,Y} = 0.$$

- We say that (X, B) is **log canonical / LC** resp. **Kawamata log terminal / KLT** if for any prime divisor E over X, we have $\mathrm{mult}_E(\mathbf{L}) \leq 1$, resp. $\mathrm{mult}_E(\mathbf{L}) < 1$ (i.e. $\mathrm{mult}_E(\mathbf{A}) \geq / > -1$).

Singularities of the MMP

- We say that (X, B) is **canonical** (resp. **terminal**) if $\mathrm{mult}_E(\mathbf{A}_B) \geq 0$ (resp. $\mathrm{mult}_E(\mathbf{A}_B) > 0$) for all divisors E exeptional over X (in particular $\mathbf{L}_{B,Y} = f_*^{-1}B$).
- We say that $(X, S + B)$ is **PLT** if $S = \lfloor S + B \rfloor$ is a prime divisor and $\mathrm{mult}_E(\mathbf{L}_{B,Y}) < 1$ for any other prime divisor over X.

Singularities of the MMP

- The coefficients of $\mathbf{A} = \mathbf{L} - \mathbf{E}$ are a measure of the singularities of (X, B). The infimum of these coefficients is the **total discrepancy**.
- More negative total discrepancies (resp. discrepancies) correspond to more singular pairs. (≥ -1, > -1, (resp. ≥ 0, > 0) correspond to LC, KLT, (resp. canonical, terminal)).

Singularities of the MMP

- The log canonical and KLT conditions can be checked on one (any) log resolution.
- If $f : Y \to X$ is the blow up of the vertex of a cone over a rational curve of degree n with exeptional curve E, then by adjunction $-2 = (K_Y + E) \cdot E = (\nu^*(K_X) + (a_E + 1)E) \cdot E = -n(a_E + 1)$. (Since $E^2 = -n$.) Thus $a_E = -1 + \frac{2}{n}$.
- The same computation shows that if the curve is elliptic, then $a_E = -1$ and if the curve has genus $g \geq 2$, then $a_E < -1$.
- Note that if $a_E < -1$ then the discrepancy is $-\infty$.

Singularities of the MMP

- KLT singularities are rational ($R^i f_* \mathcal{O}_Y = 0$ for $i > 0$) and LC singularities are Du Bois (Kollár-Kovács).
- Terminal/canonical singularities arise from minimal/canonical models of smooth varieties X and LC/KLT singularities are the singularities of log canonical models $\mathrm{Proj}(R(K_X + B))$ where (X, B) is a SNC pair with coefficients $0 \leq b_i \leq 1/0 \leq b_i < 1$.

Singularities of the MMP

- If $a_P := \mathrm{mult}_P \mathbf{A}_B \le -1$ (resp. < -1), then we say that P is a **NKLT place** (resp. a **NLC place**) and its image $f(P)$ is a **center** of NKLT (resp. NLC) singularities.
- If (X, B) is LC and $G \ge 0$ is an $\mathbb{R}$-Cartier divisor, then the **log canonical threshold** is

$$\mathrm{lct}(X, B; G) = \sup\{c > 0 | (X, D + cG) \text{ is LC}\}.$$

- One can compute log canonical thresholds on a single log resolution (eg, $\mathrm{lct}(\mathbb{C}^2, 0; \{y^2 = x^3\}) = 5/6$).

Multiplier ideals (5 minute refresher)

- Let X be smooth, $B \geq 0$, $f : Y \to X$ a log resolution of (X, B) then the **multiplier ideal sheaf** of (X, D) is
$$\mathcal{J} = \mathcal{J}(X, B) = f_* \mathcal{O}_Y(K_{Y/X} - \lfloor f^* B \rfloor) \subset f_* \mathcal{O}_Y(K_{Y/X}) = \mathcal{O}_X.$$
- $\mathcal{J}$ is independent of the log resolution.
- $\mathcal{J} = \mathcal{O}_X$ iff (X, B) is KLT (as $K_{Y/X} - \lfloor f^* B \rfloor = \lceil \mathbf{A}_Y \rceil$).
- If B is SNC, then $\mathcal{J}(B) = \mathcal{O}_X(-\lfloor B \rfloor)$.
- If G Cartier, then $\mathcal{J}(G + B) = \mathcal{J}(B) \otimes \mathcal{O}_X(-G)$.
- $D_1 \leq D_2$ then $\mathcal{J}(D_2) \subset \mathcal{J}(D_1)$.

Multiplier ideals

- Multiplier ideals $\mathcal{J}(D)$ are a sophisticated measure of the singularities of D.
- $\mathrm{mult}_x(D) \geq n = \dim X$, then $\mathcal{J}(D) \subset \mathfrak{m}_x$ (just blow up $x \in X$).
- (Harder) If $\mathrm{mult}_x(D) < 1$, then $\mathcal{J}(D)_x = \mathcal{O}_{X,x}$.
- $\mathrm{lct}(X, 0; G) = \sup\{t | \mathcal{J}(X, tG) = \mathcal{O}_X\}$.

Nadel vanishing

An easy consequence of Kawamata-Viehweg's vanishing theorem is the following.

Theorem (Nadel vanishing)

X smooth, $f : X \to Z$ a projective morphism, $D \geq 0$ an $\mathbb{R}$-divisor, N a Cartier divisor such that $N - D$ is f-nef and f-big, then

$$R^i f_*(\mathcal{O}_X(K_X + N) \otimes \mathcal{J}(D)) = 0 \qquad \forall i > 0.$$

At first sight $\mathcal{J}(D)$ is a technical annoyance, but as we will see later it should be viewed as an opportunity.

Restrictions of multiplier ideal sheaves

- If H is a smooth divisor on a smooth variety X, $D \geq 0$ an effective $\mathbb{R}$-divisor on X whose support does not contain H. Then $\mathcal{J}(H, D|_H) \subset \mathcal{J}(X, D) \cdot \mathcal{O}_H$, and
- if $0 < s < 1$, then for all $0 < t \ll 1$ we have

$$\mathcal{J}(X, D + (1-t)H) \cdot \mathcal{O}_H \subset \mathcal{J}(H, (1-s)D|_H).$$

- This is an example of inversion of adjunction. If $\mathcal{J}(H, (1-s)D|_H) \subset \mathfrak{m}_x$ ($x \in H$ and $0 < s < 1$), then $\mathcal{J}(X, D + (1-t)H) \subset \mathfrak{m}_x$ for all $0 < t \ll 1$.

Restrictions of multiplier ideal sheaves

- (Analog for log pairs) $(X, S+B)$ an effective log pair, $\nu : S^\nu \to S$ the normalization of S and $K_{S^\nu} + B_{S^\nu} = \nu^*(K_X + S + B)$, then
  1. $(X, S+B)$ is PLT iff (S^ν, B_{S^ν}) is KLT and
  2. $(X, S+B)$ is LC iff (S^ν, B_{S^ν}) is LC.
- The first is an easy consequence of the connectedness lemma of Kollár and Shokurov, the second is a deep result of Kawakita.

Outline of the talk

Boundedness of canonical models

Theorem (Tsuji, Hacon-McKernan, Takayama)

Fix $n \in \mathbb{N}$ and $V > 0$, then there exists $m \in \mathbb{N}$ such that if X is a canonical model, $\dim X = n$ *and $K_X^n \leq V$, then mK_X is very ample. (In particular $\{K_X^n\}$ is discrete.)*

- Tsuji's idea is to first prove the following weaker result that mK_X is birational and to do this by induction on $n = \dim X$ by first proving the following weaker statements.

Birational boundedness

Theorem

Fix $n \in \mathbb{N}$, then there exists $A, B, v > 0$ such that if X is a canonical model, $\dim X = n$, *then*

1. *rK_X is birational for any $r \geq A(K_X^n)^{-1/n} + B$, and*
2. *$\mathcal{V}(n) =: \{K_X^n\}$ is discrete and $K_X^n \geq v$ for any canonical model X.*

Note that $(1)_n + (2)_n$ imply that rK_X is birational for any integer $r \geq Av^{-1/n} + B$.

Birational boundedness

- To see that $(1)_n$ implies $(2)_n$, we may assume that $K_X^n \leq V$ e.g. $K_X^n \leq 1$ and let $m = \lceil A(K_X^n)^{-1/n} + B \rceil$.
- If Z is the closure of $\phi_{|mK_X|}(X)$, then

$$\deg(Z) \leq (mK_X)^n < (A(K_X^n)^{-1/n} + B + 1)^n K_X^n < (A+B+1)^n.$$

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- Therefore Z belongs to a bounded family $\mathcal{Z} \to S$. Let $\mathcal{X}' \to \mathcal{Z}$ be a resolution of this family.
- Decomposing S in to a finite union of locally closed subsets, we may assume that $\mathcal{X}' \to S$ is a smooth morphism and $\{s \in S | \kappa(\mathcal{X}'_s) = \dim \mathcal{X}'_s\}$ is dense.
- By deformation invariance of plurigenera, all fibers $\mathcal{X}'_s$ are of general type and $\{\mathrm{vol}(K_{\mathcal{X}'_s}) | s \in S\}$ is finite and $(2)_n$ follows.

Birational boundedness

- Let $\mathcal{X} \to S$ be the relative canonical model. Then for any canonical model X with $\dim X = n$ and $\mathrm{vol}(K_X) < V$, we have $s \in S$ s.t. $X \cong \mathcal{X}_s$.
- It is easy to see that there is an integer $m \in \mathbb{N}$ such that $mK_{\mathcal{X}_s}$ is very ample for all $s \in S$ and hence mK_X is very ample for any canonical model of dimension n with $K_X^n \leq V$. (In particular $\mathrm{vol}(K_X) \geq 1/m^n$.)
- Thus, the set of all n-dimensional canonical models with volume bounded from above $0 < K_X^n \leq V$ is bounded.

Birational boundedness

- It remains to show that $(1)_{n-1} + (2)_{n-1}$ implies $(1)_n$.
- The main idea is, for any very general points $x, y \in X$, to produce a divisor $D \sim_{\mathbb{Q}} \lambda K_X$ such that
  1. $\mathcal{J}(D) \subset \mathfrak{m}_y$ and on a neighborhood of x, $\mathcal{J}(D) = \mathfrak{m}_x$, and
  2. $\lambda < m - 1 = \lceil A(K_X^n)^{-1/n} + B \rceil - 1$.

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  2. $\lambda < m - 1 = \lceil A(K_X^n)^{-1/n} + B \rceil - 1$.
- By Nadel vanishing (assuming for simplicity that X is smooth), $H^1(\omega_X^m \otimes \mathcal{J}(D)) = 0$ and hence $H^0(\omega_X^m) \to H^0(\omega_X^m \otimes \mathcal{O}_X/\mathcal{J}(D))$ is surjective.

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- Since $\mathbb{C}_x$ is a summand of $\mathcal{O}_X/\mathcal{J}(D)$, and $\mathcal{J}(D) \subset \mathfrak{m}_y$ we have produced sections vanishing at y but not at x.

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- By Nadel vanishing (assuming for simplicity that X is smooth), $H^1(\omega_X^m \otimes \mathcal{J}(D)) = 0$ and hence $H^0(\omega_X^m) \to H^0(\omega_X^m \otimes \mathcal{O}_X/\mathcal{J}(D))$ is surjective.
- Since $\mathbb{C}_x$ is a summand of $\mathcal{O}_X/\mathcal{J}(D)$, and $\mathcal{J}(D) \subset \mathfrak{m}_y$ we have produced sections vanishing at y but not at x.
- This implies that mK_X is birational.
- In what follows we will focus on the condition $\mathcal{J}(D) = \mathfrak{m}_x$ (it is not much harder to guarantee $\mathcal{J}(D) \subset \mathfrak{m}_y$.

Birational boundedness

- We have $h^0(mK_X)=\frac{m^n}{n!}K_X^n+o(m^n)$ whilst vanishing at a smooth point to order k is $k^n/n!+o(k^n)$ conditions.
- Thus we will find a divisor $D_1\sim_{\mathbb{Q}}\lambda_1 K_X$ with $\mathrm{mult}_x(D_1)\geq n$ and $\lambda_1=O((K_X^n)^{-1/n})$.
- Locally we have $\mathcal{J}(D_1)=\mathcal{I}_Z\subset\mathfrak{m}_x$ where we may assume that Z is reduced and irreducible.
- Our goal is to cut down Z to a point.

Birational boundedness

- To do this, we need to bound $(K_X|_Z)^{\dim Z}$ from below.
- Since $x \in X$ is general, Z is of general type and so by induction on the dimension $\mathrm{vol}(K_{Z'}) \geq v(d) > 0$ where $d = \dim Z$ and $Z' \to Z$ is a resolution.
- Tsuji's idea is to use Kawamata's subadjunction to compare $\mathrm{vol}(K_{Z'})$ and $\mathrm{vol}(K_X|_Z) = (K_X|_Z)^{\dim Z}$.

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Subadjunction

- Recall that $\mathcal{J}(D_1) = \mathcal{I}_Z$ near $x \in Z \subset X$ and $D_1 \sim_{\mathbb{Q}} \lambda_1 K_X$.
- Assume for simplicity that Z is smooth.
- Then, by Kawamata sub adjunction (as K_X is ample)

$$(1 + \lambda_1 + \epsilon) K_X|_Z \sim_{\mathbb{Q}} (K_X + D_1 + \epsilon K_X)|_Z \geq_{\mathbb{Q}} K_Z$$

and so $(K_X|_Z)^d \geq (\frac{1}{1+\lambda_1})^d \cdot v(d)$ where $d = \dim Z$.

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Subadjunction

- We now pick a very general point $x' \in Z$ and $D'_Z \sim_{\mathbb{Q}} \lambda' K_X|_Z$ such that $\mathrm{mult}_{x'}(D'_Z) > d$ and $\lambda' \leq d(1+\lambda_1)+1$.
- Since K_X is ample, by Serre vanishing we may assume that $D'_Z = D'|_Z$ where $D' \sim_{\mathbb{Q}} \lambda' K_X$.
- By inversion of adjunction $\mathfrak{m}_{x'} \supset \mathcal{J}((1-\delta)D_1 + (1-\eta)D') = \mathcal{I}_{Z_2}$ where $\dim Z_2 < \dim Z$.

Subadjunction

- Let $D_2 = (1-\delta)D_1 + (1-\eta)D'$ so that $D_2 \sim \lambda_2 K_X$ where $\lambda_2 = O((K_X^n)^{-1/n})$.
- Note that x' is also a very general point of X.
- Thus we may replace x, D_1, Z, λ_1 by x', D_2, Z_2, λ_2.
- After $\leq n$ iterations, we may assume that $\dim Z = 0$ i.e. that $\mathcal{J}(D) = \mathfrak{m}_x$ on a neighbourhood of $x \in X$. QED