— I —

A Theorem
**Theorem** [C. - Junyi Xie].— Let $\Gamma$ be a finite index subgroup of $\text{SL}_n(\mathbb{Z})$.

- If $\Gamma$ embeds in $\text{Bir}(\mathbb{P}^m_\mathbb{C})$ then $m \geq n - 1$.
- If $\Gamma$ embeds in $\text{Aut}(\mathbb{A}^m_\mathbb{C})$ then $m \geq n$.

**Remark.**— One can replace

- $\mathbb{P}^m_\mathbb{C}$ by a complex variety $V$ and get
  \[ \dim(V) \geq n - 1. \]
- $\text{SL}_n(\mathbb{Z})$ by an arithmetic lattice in a simple linear algebraic group $G(\mathbb{R})$ for which the congruence kernel is finite, and get
  \[ \dim(V) \geq \text{rank}_\mathbb{R}(G(\mathbb{R})). \]
Theorem [C. - Junyi Xie].— Let $\Gamma$ be a finite index subgroup of $\text{SL}_n(\mathbb{Z})$, $n \geq 3$. Let $V$ be a complex, irreducible projective variety. If $\Gamma$ embeds in $\text{Bir}(V)$ then

$$\dim(V) \geq n - 1$$

and in case of equality, $V$ is rational and the action of $\Gamma$ is conjugate to an action on $\mathbb{P}^m_\mathbb{C}$ by regular automorphisms.
Problem.— If $n > m + 1$, and $p$ is a large prime, show that

$$\text{SL}_n(\mathbb{Z}/p\mathbb{Z})$$

does not embed in $\text{Cr}_m(\mathbb{C})$.

- this is known in dimension 2 and 3 (Dolgachev-Iskovskikh, Prokhorov).
- this is not known in dimension 4.
• Assume \( \Gamma \subset \text{Aut}(\mathbb{A}_C^m) \) is finitely generated:

\[ S = \{\gamma_1, \ldots, \gamma_s\} \]

is a finite symmetric set of generators.

• Define

\[ \text{Coeff} = \{\text{coefficients of the polynomial formulae defining the } \gamma_i\} \]

• Then

**Lemma.**— *The ring \( \mathbb{Z}[\text{Coeff}] \) embeds into the ring of \( p \)-adic integers \( \mathbb{Z}_p \) (for infinitely many primes \( p \)).*
From complex to $p$-adic numbers

- $\mathbb{Z}_p = p$-adic integers
- If $a \in \mathbb{Z}$ and $a = p^s q$ with $q \wedge p = 1$, then
  $$|a|_p = p^{-s}.$$
- $\mathbb{Q}_p = \text{completion}$ of $\mathbb{Q}$ for the norm
  $$\left| \frac{a}{b} \right|_p = \frac{|a|_p}{|b|_p}$$
  and
  $$\mathbb{Z}_p = \text{closure of } \mathbb{Z}$$
  $$= \{ z \in \mathbb{Q}_p ; |z|_p \leq 1 \}$$
- Residue field $= \mathbb{Z}_p/p\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$. 
Local dynamics over $\mathbb{C}$
Local dynamics: complex case

- Consider \( f(z) = z + \epsilon z^d \), with \( d \geq 2 \) and \( \epsilon > 0 \). Then,

  One cannot find arbitrary small neighborhoods of \( 0 \in \mathbb{C} \) which are \( f \)-invariants.
Theorem [Cremer].— Consider a polynomial transformation $f : \mathbb{C} \to \mathbb{C}$ of type

$$f(z) = \lambda z + a_2 z^2 + \ldots + a_{d-1} z^{d-1} + a_d z^d$$

with $d \geq 2$. Assume that $|\lambda| = 1$ and

$$\forall \epsilon > 0, \exists q \geq 1, \text{ such that } |\lambda^q - 1| \leq \epsilon^{dq}.$$ 

Then $f$ has periodic points which are arbitrarily close to the origin $0 \in \mathbb{C}$.

Remark.— $|\lambda^q - 1| = |e^{2\pi i q \theta} - 1| \simeq 2\pi \text{ dist}(q \theta, \mathbb{Z})$. 
Sketch of Proof.–

- Assume \( f \) is a monic polynomial (i.e. \( a_d = 1 \)).
- Equation for periodic points

\[
  f^q(z) = z
\]

i.e.

\[
  z^{dq} + \ldots + (\lambda^q - 1)z = 0.
\]

One root is \( z = 0 \), the other roots \( z_i \) satisfy

\[
  \prod_{i=1}^{dq-1} z_i = \lambda^q - 1.
\]

If \( |\lambda^q - 1| < \epsilon^{dq-1} \), at least one root \( z_i \) satisfies

\[
  |z_i| < \epsilon.
\]
Local dynamics over $\mathbb{Q}_p$
- $\| \cdot \| = \text{Tate norm on } \mathbb{Z}_p[x] = \mathbb{Z}_p[x_1, \ldots, x_m]$. If

$$f(x) = \sum_{J=(j_1, \ldots, j_m)} a_J x^J$$

then

$$\| f \| = \max_J |a_J|_p.$$ 

- **Tate Algebra** $= \mathbb{Z}_p\langle x \rangle = \text{completion of } \mathbb{Z}_p[x] \text{ for } \| \cdot \|$. 

- elements of $\mathbb{Z}_p\langle x \rangle$ are **power series**

$$f(x) = \sum_{J=(j_1, \ldots, j_m)} a_J x^J$$

with $|a_J|_p \to 0$ as $|J| = j_1 + \ldots + j_m$ goes to $+\infty$. 
• The **polydisk** of dimension $m$ is $\mathcal{U} = \mathbb{Z}_p^m$.

• Analytic endomorphisms (or Tate endomorphisms) are maps

\[
 f : \mathcal{U} \to \mathcal{U} \\
 x \mapsto (f_1(x), \ldots, f_m(x))
\]

where each $f_i$ is in $\mathbb{Z}_p\langle x \rangle$.

• **Analytic diffeomorphisms** (or Tate diffeomorphisms) form a group

\[
 \text{Diff}^{\text{Tate}}(\mathcal{U}).
\]

• Write $f(x) \equiv \text{Id}(x) \pmod{p^c}$ if all the coefficients $b_J$ in the power series expansion of $f(x) - \text{Id}(x)$ satisfy

\[
 b_J \in p^c\mathbb{Z}_p \quad (i.e. |b_J|_p \leq p^{-c}).
\]
Theorem [J. Bell, B. Poonen].— Let $f$ be an analytic endomorphism of the polydisk $\mathcal{U}$. Assume
\[ f \equiv \text{Id} \pmod{p^c} \text{ with } c > 1/(p - 1). \]

There exists a Tate analytic map
\[ \Phi: \mathbb{Z}_p \times \mathcal{U} \to \mathcal{U} \]

such that
- $\Phi$ determines an analytic action of $(\mathbb{Z}_p, +)$ on $\mathcal{U}$.
- $\Phi(n, x) = f^n(x)$ for all $n \in \mathbb{Z}$.

In particular, $f$ is a diffeomorphism of $\mathcal{U}$, with $f^{-1}(x) = \Phi(-1, x)$. 
Proof of Bell Theorem after Poonen

- Consider the operator

$$\Delta_f : h(x) \mapsto h \circ f(x) - h(x).$$

Since $f(x) \equiv \text{Id}(x) \pmod{p^c}$, one gets

$$\| \Delta_f^j(h) \| \leq p^{-jc} \| h \|$$

- The series

$$\Phi(t, x) = \sum_{j \geq 0} \binom{t}{j} \Delta_f^j(\text{Id}(x))$$

$$= \sum_{j \geq 0} \frac{t \cdot (t - 1) \cdots (t - j + 1)}{1 \cdot 2 \cdots j} \Delta_f^j(\text{Id}(x))$$

converges because

$$p\text{-adic valuation}(j!) = \left[\frac{j}{p}\right] + \left[\frac{j}{p^2}\right] + \cdots \leq \frac{j}{p - 1}$$
Proof of Bell Theorem after Poonen

Moreover, for $n \in \mathbb{Z}_+$,

$$\Phi(n, x) = \left( \sum_{j=0}^{n} \binom{n}{j} \Delta_f^j \right) (\text{Id}(x))$$

$$= (\text{Id} + \Delta_f)^n(\text{Id}(x))$$

$$= f^n(x)$$

From this relation one gets

$$\Phi(n, \Phi(n', x)) = \Phi(n + n', x)$$

for all $n$ and $n'$ in $\mathbb{Z}_p$. 
Bell Theorem, and vector fields

- Bell theorem says:
  - Start with an analytic diffeomorphism $f \in \text{Diff}^{Tate}(U)$ with $f \equiv \text{Id} \pmod{p}$
  - The action of $(\mathbb{Z}, +)$ generated by $f$ extends to an analytic action of
    
    \[
    \mathbb{Z}_p = \text{pro-}p \text{ completion of } \mathbb{Z} \\
    = \text{projective limit of the } \mathbb{Z}/p^\ell \mathbb{Z}. \\
    = \text{a one dimensional } p\text{-adic analytic group}
    \]

- The vector field
  
  \[
  \mathbf{v}(x) = \frac{\partial \Phi(t, x)}{\partial t} \bigg|_{t=0}
  \]

  is an analytic vector field whose flow is given by $\Phi(t, x)$. 
— IV —

Bell Theorem for $SL_n(Z)$
The congruence subgroup property

- **Principal congruence subgroups** of $\text{SL}_n(\mathbb{Z})$ are kernels of
  
  $$\text{SL}_n(\mathbb{Z}) \rightarrow \text{SL}_n(\mathbb{Z}/q\mathbb{Z})$$

  ($q$ a positive integer).

- $\Gamma < \text{SL}_n(\mathbb{Z})$ is a **congruence subgroup** if it contains a principal congruence subgroup.

**Theorem** [Bass, Milnor, Serre; Mennicke].— If $n \geq 3$, the group $\text{SL}_n(\mathbb{Z})$ satisfies the congruence subgroup property: every finite index subgroup of $\text{SL}_n(\mathbb{Z})$ is a congruence subgroup.

- **Consequence**: up to finite index pro-$p$ completion of $\Gamma = \text{open subgroup of } \text{SL}_n(\mathbb{Z}_p)$
Theorem [C., J. Xie].— Fix a prime $p \geq 3$. Let $\Gamma$ be a finite index subgroup of $\text{SL}_n(\mathbb{Z})$. Assume $\Gamma$ acts by analytic diffeomorphisms on $\mathcal{U} = \mathbb{Z}_p^m$ and

$$\forall f \in \Gamma, \quad f(x) \equiv \text{Id}(x) \pmod{p}.$$ 

Then the action of $\Gamma$ on $\mathcal{U}$ extends to an analytic action of the $p$-adic group $\overline{\Gamma} \subset \text{SL}_n(\mathbb{Z}_p)$ on $\mathcal{U}$.

Here,

$$\overline{\Gamma} = \text{open subgroup of } \text{SL}_n(\mathbb{Z}_p).$$

and one gets

$$\Phi: \overline{\Gamma} \times \mathcal{U} \to \mathcal{U}.$$
• Setting
  • $\Gamma$ is a finite index subgroup in $\text{SL}_n(\mathbb{Z})$.
  • $\Gamma$ acts faithfully on $\mathbb{A}^m_{\mathbb{C}}$.
  • $\Gamma$ acts faithfully on $\mathbb{A}^m_{\mathbb{Q}_p}$ by polynomial automorphisms (which are defined over $\mathbb{Z}_p$), for some prime $p \geq 3$.

• Changing $\Gamma$ in a finite index subgroup,

$$f(x) = A_0 + A_1(x) + \sum_{j \geq 2} A_j(x)$$

with

$$A_0 = 0 \pmod{p^2} \quad \text{and} \quad A_1 = \text{Id} \pmod{p}.$$ 

Then

$$p^{-1}f(px) = p^{-1}A_0 + A_1(x) + \sum_{j \geq 2} p^{j-1}A_j(x)$$

$$\equiv \text{Id} \pmod{p}$$
• Thus, an open subgroup of $\text{SL}_n(\mathbb{Z}_p)$ acts analytically on $\mathcal{U} = \mathbb{Z}_p^m$.

• Taking derivative:
The Lie algebra of $\text{SL}_n(\mathbb{Q}_p)$ embeds into the Lie algebra of Tate analytic vector fields on $\mathcal{U}$.

• Take $z_0$ a generic point of $\mathcal{U}$: the subset of vector fields vanishing at $z_0$ forms a Lie algebra of codimension $\leq m$ in $\mathfrak{sl}_n(\mathbb{Q}_p)$.

• Lie theory: $n - 1 \leq m$ (classification of maximal subalgebras in $\mathfrak{sl}_n$)

• Jacobian determinant $= 1$ implies $n \leq m$
Comments
• Alternative proof in dimension 2
  • blow-up all points of $\mathbb{P}^2$ and take the limit of Néron-Severi groups.
  • classes with self-intersection $1 = \text{an infinite dimensional symmetric space } = \mathbb{H}^\infty$, the infinite dimensional hyperbolic space.
  • an action of $\text{SL}_n(\mathbb{Z})$ on such a space has a fixed point if $n \geq 3$ (Kazhdan; Faraut and Harzallah)
  • a fixed point gives an invariant polarization.
• Alternative proof for regular automorphisms of complex projective varieties
  • $\Gamma$ acts on the cohomology of $V$.
  • for $n \geq 3$, Margulis super-rigidity implies that the action extends to an action of $\text{SL}_n(\mathbb{R})$.
  • Hodge index theorem and representation theory imply $\dim(V) \geq n$ if the action on cohomology is not trivial.