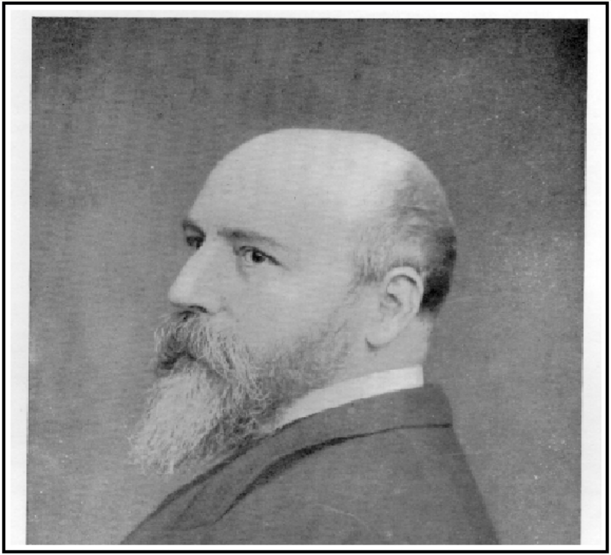
Luigi Cremona (1830 – 1903)



— I —

A Theorem

Theorem [C. - Junyi Xie].— *Let* Γ *be a finite index subgroup of* $\mathsf{SL}_n(\mathbf{Z})$.

- *If* Γ *embeds in* $\mathrm{Bir}(\mathbb{P}^m_{\mathbf{C}})$ *then* $m \geq n-1$.
- *If* Γ *embeds in* $\mathrm{Aut}(\mathbb{A}^m_{\mathbf{C}})$ *then* $m \geq n$.

Remark.— One can replace

- $\mathbb{P}^m_{\mathbf{C}}$ by a complex variety V and get

$$\dim(V) \geq n-1.$$

- $\mathsf{SL}_n(\mathbf{Z})$ by an arithmetic lattice in a simple linear algebraic group $G(\mathbf{R})$ for which the congruence kernel is finite, and get

$$\dim(V) \geq \mathrm{rank}_{\mathbf{R}}(G(\mathbf{R})).$$

Theorem [C. - Junyi Xie].— *Let* Γ *be a finite index subgroup of* $\mathrm{SL}_n(\mathbf{Z})$, $n \geq 3$. *Let* V *be a complex, irreducible projective variety. If* Γ *embeds in* $\mathrm{Bir}(V)$ *then*

$$\dim(V) \geq n - 1$$

and in case of equality, V *is rational and the action of* Γ *is conjugate to an action on* $\mathbb{P}^m_{\mathbf{C}}$ *by regular automorphisms.*

Problem.— *If $n > m+1$, and p is a large prime, show that*

$$\mathsf{SL}_n(\mathbf{Z}/p\mathbf{Z})$$

does not embed in $\mathsf{Cr}_m(\mathbf{C})$.

- this is known in dimension 2 and 3 (Dolgachev-Iskovskikh, Prokhorov).
- this is not known in dimension 4.

- Assume $\Gamma \subset \mathrm{Aut}(\mathbb{A}^m_{\mathbf{C}})$ is finitely generated:

$$S = \{\gamma_1, \ldots, \gamma_s\}$$

is a finite symmetric set of generators.

- Define

$$Coeff = \{\text{coefficients of the polynomial formulae defining the } \gamma_i\}$$

- Then

Lemma.— *The ring $\mathbf{Z}[Coeff]$ embedds into the ring of p-adic integers $\mathbf{Z}_p$ (for infinitely many primes p).*

- $\mathbf{Z}_p = p$-adic integers
- If $a \in \mathbf{Z}$ and $a = p^s q$ with $q \wedge p = 1$, then

$$|a|_p = p^{-s}.$$

- $\mathbf{Q}_p =$ **completion** of $\mathbf{Q}$ for the norm

$$\left|\frac{a}{b}\right|_p = \frac{|a|_p}{|b|_p}$$

and

$$\begin{aligned} \mathbf{Z}_p &= \text{closure of } \mathbf{Z} \\ &= \{z \in \mathbf{Q}_p \,;\, |z|_p \leq 1\} \end{aligned}$$

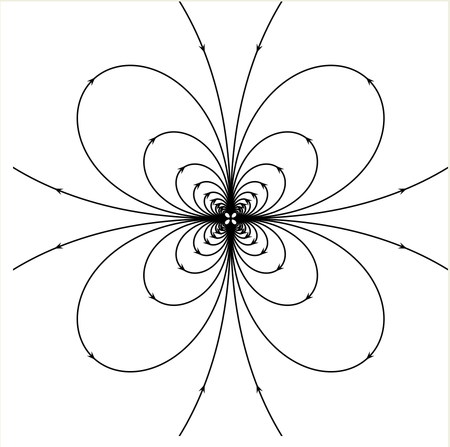
- Residue field $= \mathbf{Z}_p / p\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z} = \mathbf{F}_p$.

— II —

Local dynamics over $\mathbb{C}$

- Consider $f(z) = z + \epsilon z^d$, with $d \geq 2$ and $\epsilon > 0$. Then,

  One cannot find arbitrary small neighborhoods of $0 \in \mathbf{C}$ *which are* f*-invariants.*



Theorem [Cremer].— *Consider a polynomial transformation* $f : \mathbf{C} \to \mathbf{C}$ *of type*

$$f(z) = \lambda z + a_2 z^2 + \ldots + a_{d-1} z^{d-1} + a_d z^d$$

with $d \geq 2$. *Assume that* $|\lambda| = 1$ *and*

$$\forall \epsilon > 0, \exists q \geq 1, \text{ such that } |\lambda^q - 1| \leq \epsilon^{d^q}.$$

Then f *has* **periodic points** *which are* **arbitrarily close to the origin** $0 \in \mathbf{C}$.

Remark.— $|\lambda^q - 1| = |e^{2\pi i q\theta} - 1| \simeq 2\pi \operatorname{dist}(q\theta, \mathbf{Z})$.

Sketch of Proof.–

- Assume f is a monic polynomial (i.e. $a_d = 1$).
- Equation for periodic points

$$f^q(z) = z$$

i.e.

$$z^{d^q} + \ldots + (\lambda^q - 1)z = 0.$$

One root is $z = 0$, the other roots z_i satisfy

$$\prod_{i=1}^{d^q-1} z_i = \lambda^q - 1.$$

If $|\lambda^q - 1| < \epsilon^{d^q-1}$, at least one root z_i satisfies

$$|z_i| < \epsilon.$$

— III —

Local dynamics over $\mathbf{Q}_p$

- $\| \cdot \| =$ Tate norm on $\mathbf{Z}_p[\mathbf{x}] = \mathbf{Z}_p[x_1, \ldots, x_m]$. If

$$f(\mathbf{x}) = \sum_{J=(j_1,\ldots,j_m)} a_J \mathbf{x}^J$$

then

$$\| f \| = \max_J |a_J|_p.$$

- **Tate Algebra** $= \mathbf{Z}_p\langle \mathbf{x} \rangle =$ completion of $\mathbf{Z}_p[\mathbf{x}]$ for $\| \cdot \|$.
- elements of $\mathbf{Z}_p\langle \mathbf{x} \rangle$ are **power series**

$$f(\mathbf{x}) = \sum_{J=(j_1,\ldots,j_m)} a_J \mathbf{x}^J$$

with $|a_J|_p \to 0$ as $|J| = j_1 + \ldots + j_m$ goes to $+\infty$.

- The **polydisk** of dimension m is $\mathcal{U} = \mathbf{Z}_p^m$.
- Analytic endomorphisms (or Tate endomorphisms) are maps

$$\begin{aligned} f \quad : \quad & \mathcal{U} \to \mathcal{U} \\ & \mathbf{x} \mapsto (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})) \end{aligned}$$

where each f_i is in $\mathbf{Z}_p\langle \mathbf{x} \rangle$.

- **Analytic diffeomorphisms** (or Tate diffeomorphisms) form a group

$$\mathrm{Diff}^{\,Tate}(\mathcal{U}).$$

- Write $f(\mathbf{x}) \equiv \mathrm{Id}(\mathbf{x}) \pmod{p^c}$ if all the coefficients b_J in the power series expansion of $f(\mathbf{x}) - \mathrm{Id}(\mathbf{x})$ satisfy

$$b_J \in p^c \mathbf{Z}_p \quad (i.e. |b_J|_p \leq p^{-c}).$$

Theorem [J. Bell, B. Poonen].— *Let f be an analytic endomorphism of the polydisk* $\mathcal{U}$*. Assume*

$$f \equiv Id \pmod{p^c} \text{ with } \quad c > 1/(p-1).$$

There exists a Tate analytic map

$$\Phi \colon \mathbf{Z}_p \times \mathcal{U} \to \mathcal{U}$$

such that

- Φ *determines an analytic action of* $(\mathbf{Z}_p, +)$ *on* $\mathcal{U}$*.*
- $\Phi(n, \mathbf{x}) = f^n(\mathbf{x})$ *for all* $n \in \mathbf{Z}$

In particular, f is a diffeomorphism of $\mathcal{U}$*, with* $f^{-1}(\mathbf{x}) = \Phi(-1, \mathbf{x})$.

- Consider the operator

$$\Delta_f : h(\mathbf{x}) \mapsto h \circ f(\mathbf{x}) - h(\mathbf{x}).$$

Since $f(\mathbf{x}) \equiv \mathrm{Id}(\mathbf{x}) \pmod{p^c}$, one gets

$$\| \Delta_f^j(h) \| \leq p^{-jc} \| h \|$$

- The series

$$\begin{aligned} \Phi(t, \mathbf{x}) &= \sum_{j \geq 0} \binom{t}{j} \Delta_f^j(\mathrm{Id}(\mathbf{x})) \\ &= \sum_{j \geq 0} \frac{t \cdot (t-1) \cdots (t-j+1)}{1 \cdot 2 \cdots j} \Delta_f^j(\mathrm{Id}(\mathbf{x})) \end{aligned}$$

converges because

$$p\text{-adic valuation}(j!) = [\frac{j}{p}] + [\frac{j}{p^2}] + \ldots \leq \frac{j}{p-1}$$

- Moreover, for $n \in \mathbf{Z}_+$,

$$\begin{aligned}
\Phi(n,\mathbf{x}) &= \left(\sum_{j=0}^{n} \begin{pmatrix} n \\ j \end{pmatrix} \Delta_f^j\right)(\mathrm{Id}(\mathbf{x})) \\
&= (Id + \Delta_f)^n(\mathrm{Id}(\mathbf{x})) \\
&= f^n(\mathbf{x})
\end{aligned}$$

- From this relation one gets

$$\Phi(n, \Phi(n', \mathbf{x})) = \Phi(n + n', \mathbf{x})$$

for all n and n' in $\mathbf{Z}_p$.

- Bell theorem says :
  - Start with an analytic diffeomorphism $f \in \mathrm{Diff}^{Tate}(\mathcal{U})$ with $f \equiv \mathrm{Id} \pmod{p}$
  - The action of $(\mathbf{Z}, +)$ generated by f extends to an analytic action of

$$\begin{aligned} \mathbf{Z}_p &= \text{pro-}p \text{ completion of } \mathbf{Z} \\ &= \text{projective limit of the } \mathbf{Z}/p^\ell\mathbf{Z}. \\ &= \text{a one dimensional } p\text{-adic analytic group} \end{aligned}$$

- The vector field

$$\mathbf{v}(\mathbf{x}) = \frac{\partial \Phi(t, \mathbf{x})}{\partial t}_{|t=0}$$

is an analytic vector field whose flow is given by $\Phi(t, \mathbf{x})$.

— IV—

Bell Theorem for $\mathrm{SL}_n(\mathbf{Z})$

- **Principal congruence subgroups** of $\mathsf{SL}_n(\mathbf{Z})$ = kernels of

$$\mathsf{SL}_n(\mathbf{Z}) \to \mathsf{SL}_n(\mathbf{Z}/q\mathbf{Z})$$

(q a positive integer).

- $\Gamma < \mathsf{SL}_n(\mathbf{Z})$ is a **congruence subgroup** if it contains a principal congruence subgroup.

Theorem [Bass, Milnor, Serre ; Mennicke].— *If $n \geq 3$, the group $\mathsf{SL}_n(\mathbf{Z})$ satisfies* **the congruence subgroup property***: every finite index subgroup of $\mathsf{SL}_n(\mathbf{Z})$ is a congruence subgroup.*

- **Consequence**: up to finite index

pro-p completion of Γ = open subgroup of $\mathsf{SL}_n(\mathbf{Z}_p)$

Theorem [C., J. Xie].— *Fix a prime* $p \geq 3$.
Let Γ *be a finite index subgroup of* $\mathrm{SL}_n(\mathbf{Z})$. *Assume* Γ *acts by analytic diffeomorphisms on* $\mathcal{U} = \mathbf{Z}_p^m$ *and*

$$\forall f \in \Gamma, \quad f(\mathbf{x}) \equiv Id(\mathbf{x}) \ (mod\ p).$$

Then the action of Γ *on* $\mathcal{U}$ *extends to an analytic action of the p-adic group* $\overline{\Gamma} \subset \mathrm{SL}_n(\mathbf{Z}_p)$ *on* $\mathcal{U}$.

Here,

$$\overline{\Gamma} = \text{open subgroup of } \mathrm{SL}_n(\mathbf{Z}_p).$$

and one gets

$$\Phi \colon \overline{\Gamma} \times \mathcal{U} \to \mathcal{U}.$$

- Setting
  - Γ is a finite index subgroup in $\mathsf{SL}_n(\mathbf{Z})$.
  - Γ acts faithfully on $\mathbb{A}^m_{\mathbf{C}}$
  - Γ acts faithfully on $\mathbb{A}^m_{\mathbf{Q}_p}$ by polynomial automorphisms (which are defined over $\mathbf{Z}_p$), for some prime $p \geq 3$.
- Changing Γ in a finite index subgroup,

$$f(\mathbf{x}) = A_0 + A_1(\mathbf{x}) + \sum_{j\geq 2} A_j(\mathbf{x})$$

with

$$A_0 = 0 \pmod{p^2} \quad \text{and} \quad A_1 = \mathrm{Id} \pmod{p}.$$

Then

$$\begin{aligned} p^{-1} f(p\mathbf{x}) &= p^{-1}A_0 + A_1(\mathbf{x}) + \sum_{j\geq 2} p^{j-1} A_j(\mathbf{x}) \\ &\equiv \mathrm{Id} \pmod{p} \end{aligned}$$

Conclusion II

- Thus, an open subgroup of $\mathrm{SL}_n(\mathbf{Z}_p)$ acts analytically on $\mathcal{U} = \mathbf{Z}_p^m$.
- Taking derivative :
The Lie algebra of $\mathrm{SL}_n(\mathbf{Q}_p)$ embeds into the Lie algebra of Tate analytic vector fields on $\mathcal{U}$.
- Take z_0 a generic point of $\mathcal{U}$: the subset of vector fields vanishing at z_0 forms a Lie algebra of codimension $\leq m$ in $\mathrm{sl}_n(\mathbf{Q}_p)$.
- Lie theory : $n - 1 \leq m$ (classification of maximal subalgebras in $\mathfrak{sl}_n$)
- Jacobian determinant $= 1$ implies $n \leq m$

— V—

Comments

Conclusion II

- Alternative proof in dimension 2
  - blow-up all points of $\mathbb{P}^2$ and take the limit of Néron-Severi groups.
  - classes with self-intersection $1 =$ an infinite dimensional symmetric space $= \mathbb{H}^\infty$, the infinite dimensional hyperbolic space.
  - an action of $\mathsf{SL}_n(\mathbf{Z})$ on such a space has a fixed point if $n \geq 3$ (Kazhdan; Faraut and Harzallah)
  - a fixed point gives an invariant polarization.

- Alternative proof for regular automorphisms of complex projective varieties
  - Γ acts on the cohomology of V.
  - for $n \geq 3$, Margulis super-rigidity implies that the action extends to an action of $\mathsf{SL}_n(\mathbf{R})$.
  - Hodge index theorem and representation theory imply $\dim(V) \geq n$ if the action on cohomology is not trivial.