Luigi Cremona (1830 – 1903)
Automorphisms and birational transformations

- \( k \) = a field
  \( V_k \) = an algebraic variety, defined over \( k \)
- \( \text{Aut}(V_k) \) = group of regular automorphisms
  \( \text{Bir}(V_k) \) = group of birational transformations
- Compare to
  - linear algebraic groups, \( \text{GL}(W) \) with \( W \) a vector space
  - groups of diffeomorphisms, \( \text{Diff}^\infty(M) \) with \( M \) a compact manifold
Cremona group in \( m \) variables

- \( \mathbb{A}^m_k = \) affine space of dimension \( m \).
  \( \text{Aut}(\mathbb{A}^m_k) = \) polynomial automorphisms
  \[ (x_1, \ldots, x_m) \mapsto (f_1, \ldots, f_m) \]
  with \( f_i \in k[x_1, \ldots, x_m] \)

- \( \text{Bir}(\mathbb{A}^m_k) = \) birational transformations :
  \[ f_i \in k(x_1, \ldots, x_m). \]
  = automorphisms of \( k(x_1, \ldots, x_m) \)
  = \( \text{Cr}_m(k) \), \textbf{Cremona group in} \( m \) \text{ variables}
  = group of \textbf{birational transformations} of \( \mathbb{P}^m_k \)

\[
\text{Bir}(\mathbb{A}^m_k) = \text{Bir}(\mathbb{P}^m_k) = \text{Cr}_m(k)
\]
From dimension 1 to dimension $m$
The Cremona group in 1 variable

• **Dimension 1 .—**
  • $\text{Aut}(\mathbb{A}^1_k) = \text{Affine group} = \{z \mapsto az + b | a \neq 0\}$
  • $\text{Cr}_1(k) = \text{linear projective transformations} = \text{PGL}_2(k)$

For $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{GL}_2(k)$

$$X \in \mathbb{A}^1_k \mapsto \frac{aX + b}{cX + d},$$

$$[x : y] \in \mathbb{P}^1_k \mapsto [ax + by : cx + dy].$$

• For $m \geq 2$, $\text{Aut}(\mathbb{A}^m_k)$ and $\text{Cr}_m(k)$ are “infinite dimensional”:

$$(X, Y) \mapsto (X, Y + p(X)), \text{ with any } p \in k[X].$$
• $\text{Cr}_m(k)$ contains $\text{PGL}_{m+1}(k)$

This simple group has rank $m$, with maximal torus

$$\Delta_m = \text{diagonal subgroup} = G_{\text{mult}}(k)^m$$

• $\text{Cr}_m(k) \cong \text{Bir}((\mathbb{P}^1_k)^m)$ contains $\text{PGL}_2(k)^m$

with the same maximal torus $\Delta_m$. 
The group $GL_m(\mathbb{Z})$ acts by **monomial transformations**

- $(X, Y) \mapsto (X^2Y, XY)$
- $(X, Y) \mapsto (X^{-1}, Y^{-1})$ i.e. $\sigma[x : y : z] = [yz : zx : xy]$

- It normalizes the torus $\Delta_m$:

$$GL_m(\mathbb{Z}) \text{ is the group of automorphisms of the algebraic group } \left(\mathbb{G}_{mult}\right)^m.$$
Theorem [Enriques; Demazure].— Let $k$ be an algebraically closed field.

- All maximal algebraic tori $\mathbb{G}_{\text{mult}}(k)^r$ in $\text{Cr}_m(k)$ are conjugate to the diagonal subgroup $\Delta_m \subset \text{PGL}_{m+1}(k)$.
- The normalizer of $\Delta_m$ is the semi direct product $\text{GL}_m(\mathbb{Z}) \rtimes \Delta_m$.

Thus, $\text{Cr}_m(k)$ has rank $m$, $\Delta_m$ is a maximal torus, and $\text{GL}_m(\mathbb{Z})$ is its Weyl group.
Indeterminacy locus

- \( \text{Ind}(f) \) = set of **indeterminacy points**
  = an algebraic subset of codimension \( \geq 2 \).
- If \( m = 2 \), \( \text{Ind}(f) \) is a finite set.
- Let \( E(x_1, \ldots, x_{m-1}) = 0 \) be the equation of a hypersurface \( W_E \subset \mathbb{A}_k^{m-1} \). Then

  \[
  f(x_1, \ldots, x_m) = (x_1, \ldots, x_{m-1}, x_m \cdot E(x_1, \ldots, x_{m-1}))
  \]

contracts \( W_E \times \mathbb{A}^1 \) onto \( W_E \times \{0\} \).
Example of blow-down
Example of blow-down
Theorem [Noether, Castelnuovo].— Let $k$ be an algebraically closed field. The Cremona group in 2 variables is generated by

- $\text{PGL}_3(k)$ and
- $\sigma[x : y : z] = [yz : zx : xy]$

Theorem [Hudson, Pan].— To generate the Cremona group $\text{Cr}_3(k)$, one needs as many families of generators as families of curves.
Idea of Proof (Hudson, Pan).— Consider $f$ and $g$ in $\text{Cr}_3(k)$.

- $f$ contracts $S_1, \ldots, S_r$
- $g$ contracts $S'_1, \ldots, S'_s$

where the surfaces $S_i$ and $S'_j$ are birationally equivalent to

$$\mathbb{P}_k^1 \times C_i \quad \text{and} \quad \mathbb{P}_k^1 \times C'_j.$$

If $f \circ g$ contracts a surface $S$, then $S$ is birationally equivalent to

$$\mathbb{P}_k^1 \times C$$

where $C$ is isomorphic to one of the $C_i$ or $C'_j$. 
Idea of Proof (continued).– Define the genus of \( h \in \text{Cr}_3(k) \) by

\[
\text{genus}(h) = \max\{\text{genus}(C)\}
\]

over all curves \( C \) such that \( h \) contracts a surface \( W \subset \mathbb{P}^3_k \) which is birationally equivalent to \( \mathbb{P}^1_k \times C \).

Then

- \( \text{genus}(f \circ g) \leq \max(\text{genus}(f), \text{genus}(g)) \)
- for every \( g \geq 0 \), there is an element \( h \in \text{Cr}_3(k) \) with \( \text{genus}(h) = g \)
- \( f \mapsto \text{genus}(f) \) is bounded if \( \text{deg}(f) \) is bounded.
Theorem [Lukakikh; Kollár and Mangolte].—

The group of birational transformations of $\mathbb{P}^2_R$ with no real indeterminacy points is dense in $\text{Diff}^\infty(\mathbb{P}^2(R))$.

In particular, there are birational transformations of $\mathbb{P}^2_R$ with rich dynamics.
Groups of Birational transformations
Groups of transformations

Definition. — A group \( \Gamma \) is a group of birational transformations if \( \Gamma \) acts faithfully by birational transformations on some algebraic variety \( V \).

- Similar definitions: Groups of linear transformations, groups of diffeomorphisms, groups of regular automorphisms.
- More precisely: Groups of birational transformations in dimension \( m \) over the field \( k \).
- Linear groups are groups of birational transformations. There are groups of regular automorphisms which are not linear.
A group $\Gamma$ is **finitely generated** if it is generated by a finite subset $S \subset \Gamma$.

$\Gamma$ is **residually finite** if for every $\gamma \in \Gamma \setminus \{1\}$ there exists a morphism $\rho$ from $\Gamma$ to a finite group $F$ with $\rho(\gamma) \neq 1_F$.

$\Gamma$ is **virtually torsion free** if $\Gamma$ contains a torsion free, finite index subgroup.
**Theorem** [Bass and Lubotzky].— Let Γ be finitely generated group. If Γ is a group of regular automorphisms, then Γ is residually finite and virtually torsion free.

**Proof of residual finiteness for** $\Gamma \subset \text{Aut}(\mathbb{A}_Q^m)$.—

Fix a finite, symmetric set of generators

$$\gamma_i \in \text{Aut}(\mathbb{A}_Q^m), \ 1 \leq i \leq s.$$ 

Let

$$\text{Coeff} = \text{set of all coefficients in the formulas defining the } \gamma_i.$$ 

Then,

$$\mathbb{Z}[\text{Coeff}] \subset \mathbb{Z}[1/N] \quad \text{for some } \ N.$$
Choose $\alpha \in \Gamma \setminus \{1\}$. There exists a point $x \in \mathbb{A}^m(\mathbb{Z}[1/N])$ such that
\[ \alpha(x) \neq x. \]

Fix a prime $p > N$ such that
\[ \alpha(x) \neq x \pmod{p}. \]

Then reduce mod $p$:
- $\mathbb{Z}[1/N]/p\mathbb{Z} = \mathbb{Z}/p\mathbb{Z}$ is a finite field with $p$ elements.
- $\mathbb{A}^m(\mathbb{Z}/p\mathbb{Z})$ contains $p^m$ points.
- One gets a morphism
\[ \rho: \Gamma \to \text{Permutations}(\mathbb{A}^m(\mathbb{Z}/p\mathbb{Z})) \]

with $\rho(\alpha) \neq 1$. 
Consider

$$\text{PSL}_3(\mathbb{Z}) \subset \text{PGL}_3(\mathbb{Q}) \subset \text{Cr}_2(\mathbb{Q})$$

and add the element

$$\sigma[x : y : z] = [yz : zx : xy]$$

to generate a subgroup $\Gamma$ of $\text{Cr}_2(\mathbb{Q})$.

**Remark**.— *For every prime $p$, every point $x$ in $\mathbb{P}^2(\mathbb{Z}/p\mathbb{Z})$ is an indeterminacy point of at least one element $\beta \in \Gamma$.*

**Question**.— *Does $\text{Cr}_m(\mathbb{C})$ satisfy Malcev and Selberg properties for all $m$?*
**Theorem.** The Cremona group $\text{Cr}_2(k)$ satisfies Tits alternative: If $\Gamma$ is a finitely generated subgroup of $\text{Cr}_2(k)$, then $\Gamma$ contains a free non-abelian subgroup or a finite index solvable subgroup.

- Tits alternative is satisfied in $\text{GL}_n(k)$;
- Tits alternative is **not** satisfied in $\text{Diff}^\infty(S^1)$.

**Open Problem:**
Does $\text{Cr}_m(\mathbb{C})$ satisfy Tits alternative for $m \geq 3$?