Stability and wall-crossing

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1. Hearts and tilting
Definition of a Torsion pair

Let $\mathcal{A}$ be an abelian category.

A torsion pair $(\mathcal{T}, \mathcal{F}) \subset \mathcal{A}$ is a pair of full subcategories such that:

(A) $\text{Hom}(T, F) = 0$ for $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

(B) for every object $E \in \mathcal{A}$ there is a short exact sequence

$$0 \rightarrow T \rightarrow E \rightarrow F \rightarrow 0$$

for some pair of objects $T \in \mathcal{T}$ and $F \in \mathcal{F}$. 

\[
\begin{array}{c|c|c}
\mathcal{A} & \mathcal{T} & \mathcal{F} \\
\end{array}
\]
**Definition of a heart**

Let $D$ be a triangulated category.

A heart $\mathcal{A} \subset D$ is a full subcategory such that:

(A) $\text{Hom}(A[j], B[k]) = 0$ for all $A, B \in \mathcal{A}$ and $j > k$.

(B) for every object $E \in D$ there is a finite filtration

$$0 = E_m \to E_{m+1} \to \cdots \to E_{n-1} \to E_n = E$$

with factors $F_j = \text{Cone}(E_{j-1} \to E_j) \in \mathcal{A}[-j]$. 

\[
\begin{array}{cccccc}
\cdots & \mathcal{A}[1] & \mathcal{A} & \mathcal{A}[-1] & \cdots
\end{array}
\]
Properties of hearts

(I) It would be more standard to say that $\mathcal{A} \subset D$ is the heart of a bounded t-structure on $D$. But any such t-structure is determined by its heart.

(II) The basic example is $\mathcal{A} \subset D^b(\mathcal{A})$.

(III) In analogy with that case we define $H^j_{\mathcal{A}}(E) := F_j[j] \in \mathcal{A}$.

(IV) $\mathcal{A}$ is automatically an abelian category.

(V) The short exact sequences in $\mathcal{A}$ are precisely the triangles in $D$ all of whose terms lie in $\mathcal{A}$.

(VI) The inclusion functor gives an identification $K_0(\mathcal{A}) \cong K_0(D)$. 

Suppose \( \mathcal{A} \subset D \) is a heart, and \( (\mathcal{T}, \mathcal{F}) \subset \mathcal{A} \) a torsion pair. We can define a new, tilted heart \( \mathcal{A}^\# \subset D \) as in the picture.

An object \( E \in D \) lies in \( \mathcal{A}^\# \subset D \) precisely if

\[
H_{\mathcal{A}}^{-1}(E) \in \mathcal{F}, \quad H_{\mathcal{A}}^{0}(E) \in \mathcal{T}, \quad H_{\mathcal{A}}^{i}(E) = 0 \quad \text{otherwise}.
\]
Example of tilting: threefold flop

\[ \mathcal{F}_+ = \langle \mathcal{O}_{C_+}(-i) \rangle_{i \geq 1}, \quad \mathcal{F}_- = \langle \mathcal{O}_{C_-}(-i) \rangle_{i \geq 2}. \]
Stable pairs as quotients in a tilt

Consider tilting \( \mathcal{A} = \text{Coh}(X) \subset D(X) \) with respect to the torsion pair

\[ \mathcal{T} = \{ E \in \text{Coh}(X) : \dim_{\mathbb{C}} \text{supp}(E) = 0 \}, \]

\[ \mathcal{F} = \{ E \in \text{Coh}(X) : \text{Hom}_X(\mathcal{O}_x, E) = 0 \text{ for all } x \in X \}. \]

Note that \( \mathcal{O}_X \in \mathcal{F} \subset \mathcal{A}^\#. \) We claim that

\[ \text{Pairs}(\beta, n) = \left\{ \begin{array}{l} \text{quotients } \mathcal{O}_X \to E \text{ in } \mathcal{A}^\# \text{ with } \text{ch}(E) = (0, 0, \beta, n) \end{array} \right\}. \]
Proof of the Claim about Stable Pairs

Given a short exact sequence in the category $\mathcal{A}^\#$

$$0 \longrightarrow J \longrightarrow \mathcal{O}_X \stackrel{f}{\longrightarrow} E \longrightarrow 0,$$

we take cohomology with respect to the standard heart $\mathcal{A} \subset D$.

$$0 \rightarrow H^0_\mathcal{A}(J) \rightarrow \mathcal{O}_X \xrightarrow{f} H^0_\mathcal{A}(E) \rightarrow H^1_\mathcal{A}(J) \rightarrow 0 \rightarrow H^1_\mathcal{A}(E) \rightarrow 0.$$ 

It follows that $E \in \mathcal{A} \cap \mathcal{A}^\# = \mathcal{F}$ and $\text{coker}(f) = H^1_\mathcal{A}(J) \in \mathcal{T}$. 

$$\begin{aligned} \cdots & \quad \mathcal{T} \quad \mathcal{F} \quad \mathcal{T}[-1] \quad \cdots \\ \mathcal{A}^\# & \quad \{ \} \\ \cdots \end{aligned}$$
(A) Hall algebras: $\text{Hall}_{\text{fty}}(C)$, $\text{Hall}_{\text{mot}}(C)$.

\[
\begin{array}{c}
0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \\
(A, C) & \quad (A, C) \\
\wedge & \wedge \\
\quad B & \quad B \\
\end{array}
\]

\[
M \times M \xleftarrow{(a,c)} M^{(2)} \xrightarrow{b} M
\]

(B) Character map $\text{ch}: K_0(C) \rightarrow N \cong \mathbb{Z}^\oplus n$.

(C) Quantum torus: $\mathbb{C}_q[N] = \bigoplus_{\alpha \in N} \mathbb{C}(t) \cdot x^\alpha$ with

\[
x^\alpha \ast x^\gamma = q^{-\frac{1}{2}(\gamma,\alpha)} \cdot x^{\alpha + \gamma}.
\]

(D) Integration map: $\mathcal{I}: \text{Hall}(C) \rightarrow \mathbb{C}_q[N]$. 
Positive cones and completions

Choosing a basis \((e_1, \cdots, e_n)\) for the group \(N\) gives an identification

\[
\mathbb{C}[N] = \mathbb{C}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}].
\]

We often need to use the positive cone

\[
N_+ = \left\{ \sum_{i=1}^{n} \lambda_i e_i : \lambda_i \geq 0 \right\} \subset N,
\]

and the associated completion

\[
\mathbb{C}[[N_+]] \cong \mathbb{C}[[x_1, \cdots, x_n]].
\]

We can similarly define the completed quantum torus \(\mathbb{C}_q[[N_+]]\).
Sketch proof of the DT/PT identity

(I) Reineke’s identity: \( \delta^\mathcal{O}_\mathcal{A} = \text{Quot}^\mathcal{O}_\mathcal{A} \ast \delta_\mathcal{A} \) and \( \delta^\mathcal{O}_{\mathcal{A}^\#} = \text{Quot}^\mathcal{O}_{\mathcal{A}^\#} \ast \delta_{\mathcal{A}^\#} \).

(II) Torsion pair identities: \( \delta_\mathcal{A} = \delta_\mathcal{T} \ast \delta_\mathcal{F} \) and \( \delta_{\mathcal{A}^\#} = \delta_\mathcal{F} \ast \delta_{\mathcal{T}[−1]} \).

(III) Torsion pair identities with sections:

\[
\delta^\mathcal{O}_\mathcal{A} = \delta^\mathcal{O}_\mathcal{T} \ast \delta^\mathcal{O}_\mathcal{F} \quad \text{and} \quad \delta^\mathcal{O}_{\mathcal{A}^\#} = \delta^\mathcal{O}_\mathcal{F} \ast \delta^\mathcal{O}_{\mathcal{T}[−1]}. 
\]

(IV) All maps \( \mathcal{O}_X \to \mathcal{T}[−1] \) are zero, so \( \delta^\mathcal{O}_{\mathcal{T}[−1]} = \delta_{\mathcal{T}[−1]} \).
Conclusion of the sketch proof

(v) Reineke’s identity again: $\delta_T^O = \text{Quot}_T^O \ast \delta_T$.

(vi) Putting it all together: $\text{Quot}_A^O \ast \delta_T = \text{Quot}_T^O \ast \delta_T \ast \text{Quot}_{A^{\#}}^O$.

(vii) Restrict to sheaves supported in dimension $\leq 1$. The Euler form is then trivial so the quantum torus is commutative. Thus

$$\mathcal{I}(\text{Quot}_A^O) = \mathcal{I}(\text{Quot}_T^O) \ast \mathcal{I}(\text{Quot}_{A^{\#}}^O).$$

(viii) Setting $t = \pm 1$ then gives the required identity

$$\sum_{\beta, n} \text{DT}(\beta, n) x^\beta y^n = \sum_n \text{DT}(0, n) y^n \cdot \sum_{\beta, n} \text{PT}(\beta, n) x^\beta y^n.$$
2. Generalized DT invariants
Moduli spaces of framed sheaves

Let $X$ be a Calabi-Yau threefold.

So far we have been discussing moduli spaces of objects in the category $D^b \text{Coh}(X)$ equipped with a kind of framing.

**Example**

The Hilbert scheme parameterizes sheaves $E \in \text{Coh}(X)$ equipped with a surjective map $f : \mathcal{O}_X \twoheadrightarrow E$.

(I) This framing data eliminates all stabilizer groups, so the moduli space is a scheme, and therefore has a well-defined Euler characteristic.

(II) In this context wall-crossing can be achieved by varying the t-structure on the derived category $D^b \text{Coh}(X)$. 
What about unframed DT invariants?

Fix a polarization of $X$ and a class $\alpha \in N$, and consider the stack

$$\mathcal{M}^{ss}(\alpha) = \{ E \in \text{Coh}(X) : E \text{ is semistable with } \text{ch}(E) = \alpha \}.$$

(A) In the case when $\alpha$ is primitive, and the polarization is general, this stack is a $\mathbb{C}^*$-gerbe over its coarse moduli space $\mathcal{M}^{ss}(\alpha)$, and we set

$$\text{DT}^{\text{naive}}(\alpha) = e(\mathcal{M}^{ss}(\alpha)) \in \mathbb{Z}.$$

Genuine DT invariants are defined using virtual cycles or by a weighted Euler characteristic as before.

(B) In the general case, Joyce figured out how to define invariants

$$\text{DT}^{\text{naive}}(\alpha) \in \mathbb{Q}$$

with good properties, and showed that they satisfy wall-crossing formulae as the polarization is varied.
Quantum and classical DT invariants

(A) The generating function for the quantum DT invariants is

\[ q\text{-}DT_\mu = \mathcal{I}(\mathcal{M}^{ss}(\mu) \subset \mathcal{M}) \in \mathbb{C}_q[[N_+]]. \]

(B) The generating function for the classical DT invariants is

\[ \lim_{q \to 1} (q - 1) \log q\text{-}DT_\mu \in \mathbb{C}[[N_+]]. \]

A difficult result of Joyce shows that this limit exists in general.

(C) The DT invariants are also encoded by the Poisson automorphism

\[ S_\mu = \exp \{ DT_\mu, - \} \in \text{Aut} \mathbb{C}[[N_+]]. \]

This coincides with the \( q = 1 \) limit of conjugation by \( q\text{-}DT_\mu \).
Suppose there is a single rigid stable bundle $E$ of slope $\mu$. Then

$$\mathcal{M}^{ss}(\mu) = \{ E^{\oplus n} : n \geq 0 \} = \bigsqcup_{n \geq 0} \text{BGL}(n, \mathbb{C}).$$

Set $\alpha = \text{ch}(E) \in \mathbb{N}$. Applying the integration map we calculate

(A) The quantum DT generating function is

$$q\text{-DT}_\mu = \sum_{n \geq 0} \frac{x^{n\alpha}}{(q^n - 1) \cdots (q - 1)} \in \mathbb{C}_q[[\mathbb{N}_+]].$$

We recognise the quantum dilogarithm $\Phi_q(x^\alpha)$. 
(B) The classical DT generating function is

\[ DT_\mu = \lim_{q \to 1} (q - 1) \cdot \log \Phi_q(x^\alpha) = \sum_{n \geq 1} \frac{x^{n\alpha}}{n^2} \]

and we conclude that \( DT(n\alpha) = 1/n^2 \).

(C) The Poisson automorphism \( S_\mu \in \text{Aut } \mathbb{C}[[N_+]] \) is

\[ S_\mu(x^\beta) = \exp \left\{ \sum_{n \geq 1} \frac{x^{n\alpha}}{n^2}, - \right\} (x^\beta) = x^\beta \cdot (1 + x^\alpha)^{\langle \alpha, \beta \rangle} \]

where the RHS should be expanded as a power series.
3. Stability conditions
Let $\mathcal{A}$ be an abelian category.

**Definition**

A stability condition on $\mathcal{A}$ is a map of groups $Z : K_0(\mathcal{A}) \to \mathbb{C}$ such that

$$0 \neq E \in \mathcal{A} \implies Z(E) \in \overline{\mathbb{H}},$$

where $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R}_{<0}$ is the semi-closed upper half-plane.
**Definitons**

(A) The phase of a nonzero object $E \in \mathcal{A}$ is

$$\phi(E) = \frac{1}{\pi} \arg Z(E) \in (0, 1],$$

(B) An object $E \in \mathcal{A}$ is $Z$-semistable if

$$0 \neq A \subset E \implies \phi(A) \leq \phi(E).$$
**Definition**

A stability condition \( Z \) has the Harder-Narasimhan property if every object \( E \in \mathcal{A} \) has a filtration

\[
0 = E_0 \subset E_1 \subset \cdots \subset E_n \subset E
\]

such that each factor \( F_i = E_i / E_{i-1} \) is \( Z \)-semistable and

\[
\phi(F_1) > \cdots > \phi(F_n).
\]

(I) If \( \mathcal{A} \) has finite length this condition is automatic.

(II) When they exist, HN filtrations are necessarily unique, because the usual argument shows that if \( F_1, F_2 \) are \( Z \)-semistable then

\[
\phi(F_1) > \phi(F_2) \implies \text{Hom}(F_1, F_2) = 0.
\]
Another Reineke identity

Let $C$ be a finitary abelian category equipped with a stability condition $Z$ having the Harder-Narasimhan property. Let

$$\delta^{ss}(\phi) \in \widehat{\text{Hall}}_{fty}(\mathcal{A})$$

be the characteristic function of $Z$-semistable objects of phase $\phi \in \mathbb{R}$.

**Lemma (Reineke)**

There is an identity $\delta_C = \prod_{\phi \in \mathbb{R}} \delta^{ss}(\phi)$.

**Proof.**

The product is taken in descending order of phase. The result follows from existence and uniqueness of the HN filtration.
Wall-crossing formula

(A) The LHS of the above identity is independent of $Z$ so given two stability conditions we get a wall-crossing formula

$$\prod_{\phi \in \mathbb{R}} \delta^{ss}(\phi, Z_1) = \prod_{\phi \in \mathbb{R}} \delta^{ss}(\phi, Z_2).$$

(B) If $\mathcal{C}$ has global dimension $\leq 1$ we can apply the integration map $\mathcal{I}$ to get an identity in the ring $\mathbb{C}_q[[\mathcal{N}_+]]$.

(C) We can then take the $q = 1$ limit and obtain an identity in the group of automorphisms of the Poisson algebra $\mathbb{C}[[\mathcal{N}^+]]$. 
Example: the $A_2$ quiver

Let $C$ be the abelian category of representations of the $A_2$ quiver. It has 3 indecomposable representations:

$$0 \rightarrow S_2 \rightarrow E \rightarrow S_1 \rightarrow 0.$$ 

We have $N = K_0(A) = \mathbb{Z}^\oplus 2 = \mathbb{Z}[S_1] \oplus \mathbb{Z}[S_2]$, 

$$\langle (m_1, n_1), (m_2, n_2) \rangle = m_2 n_1 - m_1 n_2,$$

and there are isomorphisms

$$\mathbb{C}_q[[N_+]] = \mathbb{C}\langle \langle x_1, x_2 \rangle \rangle/(x_2 * x_1 - q \cdot x_1 * x_2)$$

$$\mathbb{C}[[N_+]] = \mathbb{C}[[x_1, x_2]], \quad \{x_1, x_2\} = x_1 \cdot x_2.$$
The space $\text{Stab}(\mathcal{A})$ is isomorphic to $\mathbb{H}^2$ and there is a single wall

$$\mathcal{W} = \{ Z \in \text{Stab}(\mathcal{A}) : \text{Im} \, Z(S_2)/Z(S_1) \in \mathbb{R}_{>0} \}$$

where the object $E$ is strictly semistable.

The wall-crossing formula in $\mathbb{C}_q[[\mathcal{N}_+]]$ becomes the pentagon identity

$$\Phi_q(x_2) \ast \Phi_q(x_1) = \Phi_q(x_1) \ast \Phi_q(\sqrt{q} \cdot x_1 \ast x_2) \ast \Phi_q(x_2).$$
The semi-classical version of the wall-crossing formula is the cluster identity

\[ C_{(0,1)} \circ C_{(1,0)} = C_{(1,0)} \circ C_{(1,1)} \circ C_{(0,1)}. \]

\[ C_\alpha : x^\beta \mapsto x^\beta \cdot (1 + x^\alpha)^{\langle \alpha, \beta \rangle} \in \text{Aut } \mathbb{C}[[x_1, x_2]]. \]

It can be viewed in the group of birational automorphisms of \((\mathbb{C}^*)^2\).
4. Stability in triangulated categories
Let $D$ be a triangulated category.

**Definition**

A stability condition on $D$ is a pair $(Z, \mathcal{A})$ where

(i) $\mathcal{A} \subset D$ is a heart,

(ii) $Z : K_0(\mathcal{A}) \to \mathbb{C}$ is a group homomorphism,

such that $Z$ defines a stability condition on $\mathcal{A}$ with the HN property.

An object $E \in D$ is defined to be semistable if $E = A[n]$ for some $Z$-semistable $A \in \mathcal{A}$. The phase of $E$ is then $\phi(E) := \phi(A) + n$. 

\[ \begin{array}{ccccccc} \cdots & A[1] & \mathcal{A} & A[-1] & \cdots & D \\ \phi=2 & \phi=1 & \phi=0 & \phi=-1 & \end{array} \]
Space of stability conditions

We consider only stability conditions satisfying the extra conditions

(A) The central charge $Z : K_0(D) \to \mathbb{C}$ factors via our fixed map

$$\text{ch} : K_0(D) \to N \cong \mathbb{Z}^\oplus n.$$ 

(B) There is a $K > 0$ such that for any semistable object $E \in D$

$$Z(E) \geq K \cdot \|\text{ch}(E)\|.$$

The set $\text{Stab}(D)$ of such stability conditions has a natural topology.

Theorem

Sending a stability condition to its central charge defines a local homeomorphism

$$\text{Stab}(D) \longrightarrow \text{Hom}_\mathbb{Z}(N, \mathbb{C}) \cong \mathbb{C}^n.$$

In particular, $\text{Stab}(D)$ is a complex manifold.