Induced Representations, Intertwining Operators and Transfer

James Arthur

This paper is dedicated to George Mackey.

Abstract. Induced representations and intertwining operators give rise to distributions that represent critical terms in the trace formula. We shall describe a conjectural relationship between these distributions and the Langlands-Shelstad-Kottwitz transfer of functions

Our goal is to describe a conjectural relationship between intertwining operators and the transfer of functions. One side of the proposed identity is a linear combination of traces

$$\mathrm{tr}\big(R_P(\pi_w) \circ \mathcal{I}_P(\pi, f)\big),$$

where $\mathcal{I}_P(\pi, f)$ is the value of an induced representation at a test function f, and $R_P(\pi_w)$ is a standard self-intertwining operator for the representation. Distributions of this sort are critical terms in the trace formula. The other side is defined by the transfer of f to an endoscopic group. The endoscopic transfer of functions represents a sophisticated theory, some of it still conjectural, for comparing trace formulas on different groups [**L2**]. This is a central theme in the general study of automorphic forms.

There are a number of relatively recent ideas of Langlands, Shelstad and Kottwitz, which will have to be taken for granted in the statement of such an identity. On the other hand, induced representations and intertwining operators go back many years. They were lifelong preoccupations of George Mackey [**M1**]–[**M5**], whose investigations anticipated their future role as basic objects in modern representation theory. It is therefore fitting to devote the first part of the article to a historical introduction. I shall try to describe in elementary terms how Mackey's initial ideas are reflected in some of the ways the later theory developed. I hope that this compensates in some measure for the technical and rather sketchy nature of the remaining part of the article.

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1. Induced representations

Suppose that H is a locally compact topological group, and that S is a closed subgroup. Induction is a process that attaches a representation of H to a representation of S. George Mackey formulated the induction operation in this setting, and established many of its basic properties ([**M1**], [**M2**], [**M3**]). Induced representations now occupy a central place, often implicit, in much of representation theory and automorphic forms.

In the case of finite groups, induction had been introduced earlier by Frobenius. Since one usually takes the representation σ of S to be irreducible, we can assume that it is finite dimensional in this case. The induced representation $\mathcal{I}_S(\sigma)$ of the finite group H then acts on the finite dimensional vector space $\mathcal{H}_S(\sigma)$ of functions ϕ from H to V_σ, the complex vector space on which σ acts, such that

$$\phi(sx) = \sigma(s)\phi(x), \qquad s \in S,\ x \in H.$$

It is defined by right translation

$$\big(\mathcal{I}_S(\sigma, y)\phi\big)(x) = \phi(xy), \qquad \phi \in \mathcal{H}_S(\sigma),\ x, y \in H.$$

We can assume that σ is unitary with respect to a Hermitian inner product $(\cdot,\cdot)_\sigma$ on V_σ. Then $\mathcal{I}_S(\sigma)$ is unitary with respect to the inner product

$$(\phi_1, \phi_2) = |S\backslash H|^{-1} \sum_{x \in S\backslash H} \big(\phi_1(x), \phi_2(x)\big)_\sigma$$

on $\mathcal{H}_S(\sigma)$.

The general case is more complicated. For example, it was not clear how to define $\mathcal{I}_S(\sigma)$ so that it would be unitary if σ were unitary. Mackey dealt with this problem and others by introducing the notion of a quasi-invariant measure on $S\backslash H$.

Let $\delta_H(x)$ be the modular function of H, namely the Radon-Nikodym derivative of the right Haar measure $d_r x$ on H with respect to the left Haar measure $d_\ell x$. It is known that there is a right invariant Borel measure on $S\backslash H$ if and only if the restriction of δ_H to S equals δ_S. This condition fails in many important cases. But according to Mackey, one can always define a quasi-invariant measure $d_q x$ on $S\backslash H$. One first chooses a positive Borel function q on H that extends $\delta_S \delta_H^{-1}$, in the sense that

$$q(sx) = \delta_S(s)\delta_H(s)^{-1} q(x), \qquad s \in S,\ x \in H.$$

The measure is then defined by the condition that

$$\int_{S\backslash H} \Big(\int_S \phi(sx) d_r s \Big) d_q x = \int_H \phi(x) q(x) d_r x, \tag{1.1}$$

for any $\phi \in C_c(H)$. *Quasi-invariant* here means that for any fixed $y \in H$, the measure

$$d_{q,y} x = d_q(xy)$$

on $S\backslash H$ is equivalent to $d_q x$, in the sense that either measure is absolutely continuous with respect to the other. The Radon-Nikodym derivative of $d_{q,y}x$ with respect to $d_q y$ is given explicitly as

$$d_{q,y} x = q_y(x) d_q x,$$

for the function

$$q_y(x) = q(xy) q(x)^{-1}, \qquad x \in S\backslash H,$$

on $S\backslash H$.

Suppose that σ is an irreducible unitary representation of S on a Hilbert space V_σ. Given q, we take $\mathcal{H}_S(\sigma)$ to be the Hilbert space of Borel functions ϕ from H to V_σ such that

$$\phi(sx) = \sigma(s)\phi(x), \qquad s \in S,\ x \in H,$$

and

$$\|\phi\|_2^2 = \int_{S\backslash H} \|\phi(x)\|_\sigma^2 d_q x < \infty.$$

We then set

$$\big(\mathcal{I}_S(\sigma, y)\phi\big)(x) = \phi(xy)q_y(x)^{\frac{1}{2}}, \qquad \phi \in \mathcal{H}_S(\sigma).$$

It follows easily from the definitions that $\mathcal{I}_S(\sigma)$ is a unitary representation of H. Moreover, if q is replaced by another function q', the corresponding induced representations are unitarily equivalent. Mackey's construction is therefore essentially independent of the choice of q.

We shall be concerned here with the case that $H = G(F)$, where G is a connected reductive algebraic group over a local field F of characteristic 0. This is the setting for the local harmonic analysis developed by Harish-Chandra, with which he eventually established the Plancherel formula for $G(F)$ [**H1**], [**H2**], [**W**]. We take S to be the subgroup $P(F)$ of $G(F)$, where P is a parabolic subgroup of G with fixed Levi decomposition $P = MN_P$ over F. Since the unipotent radical N_P of P is normal in P, any unitary representation π of the Levi component $M(F)$ pulls back to $P(F)$, and can thereby be regarded as a representation of $P(F)$. Recall that we have the real vector space

$$\mathfrak{a}_M = \mathrm{Hom}\big(X(M)_F, \mathbb{R}\big)$$

attached to M, and the familiar homomorphism

$$H_M :\ M(F) \longrightarrow \mathfrak{a}_M$$

defined by

$$\big\langle H_M(m), \chi\big\rangle = \log|\chi(m)|, \qquad m \in M(F),\ \chi \in X(M)_F.$$

It allows us to form the twist

$$\pi_\lambda(m) = \pi(m)e^{\lambda(H_M(m))}, \qquad m \in M(F),\ \lambda \in \mathfrak{a}^*_{M,\mathbb{C}},$$

of π by any complex-valued linear form λ on $\mathfrak{a}_M$.

The reductive group $G(F)$ is unimodular. However $P(F)$ is not (so long as it is proper in $G(F)$). Its modular function equals

$$\delta_P(mn) = e^{2\rho_P(H_M(m))}, \qquad m \in M(F),\ n \in N_P(F),$$

where $\rho_P \in \mathfrak{a}^*_M$ is the real linear form defined by the usual half-sum of positive roots (counted with multiplicity). Mackey's induction depends on an extension q of the linear form

$$\delta_P(p)\delta_G(p)^{-1} = \delta_P(p), \qquad p \in P(F),$$

to $G(F)$. This is provided by a suitable choice of maximal compact subgroup of $G(F)$.

Let K be a fixed maximal compact subgroup of $G(F)$ such that $G(F) = P(F)K$. Given K, we define a continuous function

$$H_P :\ G(F) \longrightarrow \mathfrak{a}_M$$

by setting

$$H_P(mnk) = H_M(m), \qquad m \in M(F),\ n \in N_P(F),\ k \in K.$$

We then set

$$q(x) = e^{2\rho_P(H_P(x))}, \qquad x \in G(F).$$

Using this to define Mackey's quasi-invariant measure $d_q x$ as above, we obtain the induced representation $\mathcal{I}_{P(F)}(\pi)$ on the Hilbert space $\mathcal{H}_{P(F)}(\pi)$. (In following what seemed to be the most logical notation, we have ended up with three uses of the symbol H: the locally compact group $H = G(F)$, the function H_P, and the Hilbert space $\mathcal{H}_{P(F)}(\pi)$. I hope that this does not cause confusion.)

In this setting, it is customary to work with functions on K rather than Mackey's space of $P(F)$-equivariant functions on $G(F)$. More precisely, if π is a unitary representation of $M(F)$ on a Hilbert space V_π, let ρ_π be the restriction mapping from $\mathcal{H}_{P(F)}(\pi)$ to a space of V_π valued functions on K. In the special case that π is the trivial one-dimensional representation, ρ_π transforms the quasi-invariant measure $d_q x$ on $P(F)\backslash G(F)$ to a Haar measure dk on K (or rather, its projection to $P(F)\cap K\backslash K$). This follows from the fact that if $d_q x$ is replaced by dk on the left hand side of (1.1), each side of this identity represents the linear form on $C_c\big(G(F)\big)$ defined by a *left* Haar measure on $G(F)$. In general, ρ_π is an isometric isomorphism from $\mathcal{H}_{P(F)}(\pi)$ onto the Hilbert space $\mathcal{H}_P(\pi)$ of functions ϕ from K to V_π such that

$$\phi(pk) = \pi(p)\phi(k), \qquad p \in P(F)\cap K,\ k \in K,$$

and

$$\|\phi\|_2^2 = \int_K \|\phi(k)\|^2 dk < \infty.$$

The conjugate

$$\mathcal{I}_P(\pi, y) = \rho_\pi \circ \mathcal{I}_{P(F)}(\pi, y) \circ \rho_\pi^{-1}, \qquad y \in G(F),$$

is then a unitary representation of $G(F)$ on $\mathcal{H}_P(\pi)$. The advantage of this formalism is that the space $\mathcal{H}_P(\pi)$ remains the same under twists π_λ of π. In particular, we can regard $\mathcal{I}_P(\pi_\lambda, y)$ as an entire function of $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$ with values in the fixed Hilbert space $\mathcal{H}_P(\pi)$. As a function of y, it is a nonunitary representation of $G(F)$ if λ does not lie in the imaginary subspace $i\mathfrak{a}^*_M$ of $\mathfrak{a}^*_{M,\mathbb{C}}$.

2. Intertwining operators

Mackey was also concerned with the analysis of intertwining operators between induced representations $\mathcal{I}_S(\sigma)$ and $\mathcal{I}_{S'}(\sigma')$ of H. These are operators

$$J(\sigma', \sigma):\ \mathcal{H}_S(\sigma) \longrightarrow \mathcal{H}_{S'}(\sigma')$$

such that

$$J(\sigma', \sigma)\mathcal{I}_S(\sigma) = \mathcal{I}_{S'}(\sigma')J(\sigma', \sigma).$$

They are obviously an important part of any study of induced representations. The space of self-intertwining operators of $\mathcal{I}_S(\sigma)$, for example, governs the decomposition of this representation into irreducible constituents.

Suppose again that H is a *finite group*. In this case, Mackey gave a complete description of the vector space of intertwining operators between two induced representations $\mathcal{I}_S(\sigma)$ and $\mathcal{I}_{S'}(\sigma')$. Consider a function

$$A:\ H \longrightarrow \mathrm{Hom}(V_\sigma, V_{\sigma'})$$

such that

$$A(s'xs) = \sigma'(s')A(x)\sigma(s), \qquad s \in S,\ s' \in S'. \tag{2.1}$$

If ϕ is any function in $\mathcal{H}_S(\sigma)$, set

$$(J_A\phi)(x) = \sum_{u \in S\backslash H} A(xu^{-1})\phi(u), \qquad x \in H.$$

It is clear from the definitions that the summand is indeed invariant under left translation of u by S, that $J_A\phi$ belongs to $\mathcal{H}_{S'}(\sigma')$ as a function of x, and that as a linear transformation from $\mathcal{H}_S(\sigma)$ to $\mathcal{H}_{S'}(\sigma)$, J_A satisfies

$$J_A\mathcal{I}_S(\sigma, y) = \mathcal{I}_{S'}(\sigma', y)J_A, \qquad y \in H.$$

Mackey's result is as follows:

PROPOSITION 2.1 (Mackey[**M1**]). *The correspondence*

$$A \longrightarrow J_A$$

is an isomorphism from the vector space of linear transformations that satisfy (2.1) to the vector space of intertwining operators from $\mathcal{I}_S(\sigma)$ to $\mathcal{I}_{S'}(\sigma')$.

The proof of the proposition is elementary. Despite its simplicity, however, the correspondence $A \to J_A$ can be seen as a model for later work that is now at the heart of representation theory. This includes Harish-Chandra's theory of the Eisenstein integral, an important part of local harmonic analysis, Langlands' theory of Eisenstein series, which led to the general theory of automorphic forms, and the work of a number of people on the intertwining operators among the induced representations $\{\mathcal{I}_P(\pi)\}$. It is the last of these three topics that will be our concern here.

The domain of Mackey's correspondence is easy to characterize. As a vector space, it is a direct sum of subspaces indexed by the set of double cosets $S'\backslash H/S$. The subspace attached to a given double coset $S'wS$ is of course the space of functions A with support on $S'wS$. It is isomorphic to the space of linear operators $A(w)$ from V_σ to $V_{\sigma'}$ which have the intertwining property

$$A(w)\sigma(w^{-1}s'w) = \sigma'(s')A(w), \qquad s' \in S' \cap wSw^{-1}. \tag{2.2}$$

Assume therefore that A is supported on the double coset $S'wS$. Changing variables, we can write

$$(J_A\phi)(x) = \sum_u A(u^{-1})\phi(ux),$$

for a sum taken over elements u in the set

$$S\backslash(S'wS)^{-1} = S\backslash Sw^{-1}S' = w^{-1}(S' \cap wSw^{-1})\backslash S'.$$

We can therefore write

$$(J_A\phi)(x) = \sum_{s' \in (S' \cap wSw^{-1})\backslash S'} \sigma'(s')^{-1}A(w)\phi(w^{-1}s'x). \tag{2.3}$$

We return to the case that H equals the locally compact group $G(F)$. We fix parabolic subgroups $P = MN_P$ and $P' = M'N_{P'}$ of G, and irreducible unitary representations π and π' of $M(F)$ and $M'(F)$. Our interest is in the formal analogue of Mackey's formula, with $S = P(F)$, $S' = P'(F)$, $\sigma = \pi$ (inflated to $P(F)$) and $\sigma' = \pi'$ (inflated to $P'(F)$). Condition (2.2) requires an intertwining operator $A(w)$

between restrictions to $P'(F) \cap wP(F)w^{-1}$ of representations $w\pi$ and π'. There is not much that can be said here about the restriction of either π or π' to proper subgroups of $M(F)$ or $M'(F)$. Fortunately, this case is not particularly relevant to the basic problems in harmonic analysis. We therefore assume that w has the property that $M' = wMw^{-1}$. We assume also that the representation

$$(w\pi)(m') = \pi(w^{-1}m'w), \qquad m' \in M'(F),$$

is equivalent to π', and that $A(w)$ is an intertwining operator from $w\pi$ to π'. It follows from our condition on w that

$$S' \cap wSw^{-1} \backslash S' \cong N_{P'}(F) \cap wN_P(F)w^{-1} \backslash N_{P'}(F).$$

Since the restriction of π' to the space on the right is trivial, the formal analogue of (2.3) becomes the integral

$$(J_A\phi)(x) = \int_{N_{P'}(F)\cap wN_P(F)w^{-1}\backslash N_{P'}(F)} A(w)\phi(w^{-1}nx)\mathrm{d}n, \qquad \phi \in \mathcal{H}_{P(F)}(\pi).$$

Integrals of this sort play a central role in both harmonic analysis and automorphic forms. They have been studied by Kunze and Stein [**KuSt**], Knapp and Stein [**KnSt**], Harish-Chandra [**H1**],[**H2**], Langlands [**L1**],[**L2**], Shahidi [**Sha**], and others.

The problem is that the integral does not coverge. Following §1, we first form the conjugate

$$J_{P'|P}(w,\pi) = \rho_{\pi'} \circ J_A \circ \rho_\pi^{-1}$$

of J_A, in order to study the twists

$$\pi_\lambda, \qquad \lambda \in \mathfrak{a}^*_{M,\mathbb{C}},$$

of π. Then $J_{P'|P}(w,\pi_\lambda)$ is, at least formally, a linear operator from the Hilbert space $\mathcal{H}_P(\pi)$ to $\mathcal{H}_{P'}(\pi')$. If the real part of λ lies in the translate $\rho_P + (\mathfrak{a}^*_P)^+$ of the chamber in $\mathfrak{a}^*_M$ attached to P, $J_{P'|P}(w,\pi_\lambda)$ is defined by an absolutely convergent integral. One of the main results in the theory is that as a function of λ, $J_{P'|P}(w,\pi_\lambda)$ has meromorphic continuation to the complex vector space $\mathfrak{a}^*_{M,\mathbb{C}}$. (See the first three references above, and also [**A1**].)

The analytic continuation solves the problem, but it immediately leads to another difficulty. This is the fact that the meromorphic function $J_{P'|P}(w,\pi_\lambda)$ can easily have a pole at $\lambda = 0$. The solution of the supplementary problem is to normalize the operators. We are assuming that π is unitary. With this condition, one can write

$$J_{P'|P}(w,\pi_\lambda) = r_{P'|P}(w,\pi_\lambda)R_{P'|P}(w,\pi_\lambda),$$

where $r_{P'|P}(w,\pi_\lambda)$ is a meromorphic *scalar-valued* function, and $R_{P'|P}(w,\pi_\lambda)$ is *analytic* at $\lambda = 0$. We can then define $R_{P'|P}(w,\pi)$ to be the value of $R_{P'|P}(w,\pi_\lambda)$ at $\lambda = 0$. It is in this way that one makes sense of the generalization of Mackey's original formula (2.3). We note that an appropriate choice of functions $r_{P'|P}(w,\pi_\lambda)$ also makes the normalized operators $R_{P'|P}(w,\pi_\lambda)$ transitive relative to (P',P) and multiplicative relative to w, properties not shared by the original operators $J_{P'|P}(w,\pi_\lambda)$.

There are many ways to choose normalizing factors $r_{P'|P}(w,\pi_\lambda)$. However, Langlands has proposed a canonical construction [**L1**, p. 281–282], which depends only on the choice of a nontrivial additive character ψ_F for F, and is based on the local L-functions and ε-factors of the representations π. It has been shown that

Langlands' proposed normalization yields the desired properties if $F = \mathbb{R}$ [**A1**] and in some cases if F is p-adic [**Sha**]. However, local L-functions and ε-factors have not been defined for general representations of p-adic groups, so Langlands' normalization remains conjectural in this case. We assume it in what follows.

Suppose now that $P' = P$ and $\pi' = \pi$. Then w represents an element in the Weyl group

$$W(M) = \mathrm{Norm}(G, M)/M,$$

which stabilizes the equivalence class of π. Langlands' conjectural normalizing factors in this case depend only on the image of w in $W(M)$. On the other hand, the original integral $J_{P|P}(w, \pi_\lambda)$ does depend on w as an element in $G(F)$, in two ways actually, the left translation of the argument by w^{-1}, and the intertwining operator $A(w)$ from π to $w\pi$. However, if we require that

$$A(wm) = A(w)\pi(m), \qquad m \in M(F),$$

the two kinds of dependence cancel, and $J_{P|P}(w, \pi_\lambda)$ also depends on w only as an element in $W(M)$. The best way to represent this condition on $A(w)$ is to take

$$A(w) = \pi_w(w),$$

where π_w is an extension of π to a representation of the group generated by the subset $M_w(F) = M(F)w$ of $G(F)$. With this understanding, we write

$$R_P(\pi_w) = R_{P|P}(w, \pi).$$

The choice of extension π_w is of course not unique, but it does serve as a convenient way to represent the mildly noncanonical nature of the operator $R_P(\pi_w)$.

For simplicity, we shall confine ourselves to the case that π belongs to the relative discrete series $\Pi_2(M)$ of $M(F)$. Since $P' = P$ and $\pi' = \pi$, $R_P(\pi_w)$ is a self-intertwining operator of $\mathcal{I}_P(\pi)$. It tells us something about the irreducible constituents of this induced representation. In fact, a well known theorem of Harish-Chandra [**H1**, Theorem 38.1] asserts that the set

$$R_P(\pi_w), \qquad w \in W(M),$$

spans the space of all self-intertwining operators of $\mathcal{I}_P(\pi)$. A further theorem of Knapp and Stein [**KnSt**, Theorem 13.4] and Silberger [**Sil**] singles out a basis of this space in terms of the R-group of π, thereby characterizing the irreducible constituents of $\mathcal{I}_P(\pi)$.

Our interest is in a related question. Suppose that f belongs to the Hecke algebra $\mathcal{H}(G)$ of functions in $C_c^\infty\big(G(F)\big)$ that are left and right K-finite. The operator

$$\mathcal{I}_P(\pi, f) = \int_{G(F)} f(y)\mathcal{I}_P(\pi, y)\mathrm{d}y$$

on $\mathcal{H}_P(\pi)$ is then of trace class.

Problem. *Given $\pi \in \Pi_2(M)$ and $w \in W(M)$ with $w\pi \cong \pi$ as above, compute*

$$\mathrm{tr}\big(R_P(\pi_w)\mathcal{I}_P(\pi, f)\big), \qquad f \in \mathcal{H}(G), \tag{2.4}$$

The linear form (2.4) is obviously a linear combination of characters of irreducible constituents of $\mathcal{I}_P(\pi)$. The problem is to determine their coefficients. However, the problem is not well posed as stated, since the extension π_w of π is not uniquely determined. There is also the fact that Langlands' normalizing factor depends on the choice of additive character. Although these ambiguities amount only to a scalar multiple of (2.4), the matter is serious. For among other things, the scalar ambiguities will obviously proliferate as w, π and F vary.

The linear form (2.4) is the main local term on the spectral side of the global trace formula [**A5**]. It drives basic questions on how one is to interpet the automorphic discrete spectrum, especially as the underlying group varies. The theory of transfer is supposed to govern relations among representations as G varies. It is therefore natural that we turn to this theory in trying to resolve the ambiguities inherent in (2.4).

3. Transfer

The reader will note that we had stepped up the pace of discussion by the end of the last section. We shall have to do so in a more serious way here, since we cannot give much of a review of the theory of transfer. I ought really to say *endoscopic* transfer, since the heart of the theory is a correspondence $f \to f'$ from functions on G to functions on endoscopic groups G' for G. It is due to Langlands, Shelstad and Kottwitz [**L2**], [**LS**], [**KoSh**]. The expected properties of the correspondence have been established for real groups [**She**] (at least in the untwisted case), but remain largely conjectural in the p-adic case. I shall assume a familiarity with the main points of the theory, both established ones and those that are conjectural.

I recall that an endoscopic group G' for G is a quasisplit group over F, which comes with an embedding $\widehat{G}' \subset \widehat{G}$ of its complex dual group into that of G. It represents a larger structure $(G', \mathcal{G}', s', \xi')$, known as an *endoscopic datum* [**LS**, p. 224]. The group G comes with a finite set of endoscopic data, taken up to the relevant notion of isomorphism. The simplest interesting example is perhaps the case that G equals the split group $SO(2n+1)$. We take s' to be a semisimple element in $\widehat{G} = Sp(2n, \mathbb{C})$ whose centralizer in $\widehat{G}$ is a product

$$\widehat{G}' = Sp(2m, \mathbb{C}) \times Sp(2n-2m, \mathbb{C}).$$

It yields an endoscopic datum, represented by the split group

$$G' = SO(2m+1) \times SO(2n-2m+1).$$

There are no further choices to be made in this case. We take $\mathcal{G}'$ to be the L-subgroup $\widehat{G}' \times W_F$ of the Weil form of the L-group ${}^LG = \widehat{G} \times W_F$ of G, and ξ' to be the identity L-embedding of ${}^LG'$ into ${}^LG$. In this way, we obtain a set of representatives of endoscopic data for G that are *elliptic*, in the sense that the image of $\mathcal{G}'$ under ξ' in ${}^LG$ is contained in no proper parabolic subgroup.

More generally, we have the notion of a *twisted* endoscopic datum $(G', \mathcal{G}', s', \xi')$ attached to an automorphism θ of G over F [**KoSh**, p. 17]. For example, consider the standard outer automorphism $\theta(g) = {}^tg^{-1}$ of the split group $G = GL(2n+1)$. We take s' to be a semisimple element in the coset $\widehat{G} \rtimes \widehat{\theta}$ whose connected centralizer in $\widehat{G} = GL(2n+1, \mathbb{C})$ is a product

$$\widehat{G}' = SO(2m+1, \mathbb{C}) \times Sp(2n-2m, \mathbb{C}).$$

(Here $\widehat{\theta}$ is a dual automorphism, which in this case can be taken to be the automorphism in the inner class of θ that preserves the standard splitting of $GL(2n+1, \mathbb{C})$.) This gives rise to an endoscopic datum represented by the split group

$$G' = Sp(2m) \times SO(2n-2m+1).$$

We can still take $\mathcal{G}'$ to be the L-group ${}^LG' = \widehat{G}' \times W_F$, but there is a supplementary choice to be made here in the L-embedding ξ' of $\mathcal{G}'$ into ${}^LG$. This is because the full centralizer of s' in $\widehat{G}$ is the product of the full orthogonal group $O(2m+1,\mathbb{C})$ with $Sp(2n-2m,\mathbb{C})$. The choice is that of a homomorphism

$$\eta' : \; W_F \;\longrightarrow\; O(2m+1,\mathbb{C})/SO(2m+1,\mathbb{C}) = \mathbb{Z}/2\mathbb{Z},$$

which is to say a character on W_F (or equivalently $\Gamma_F = \mathrm{Gal}(\overline{F}/F)$) with $(\eta')^2 = 1$. It allows us to take ξ' to be the L-embedding

$$g' \times w \;\longrightarrow\; g'\eta(w) \times w, \qquad g' \in \widehat{G}',\; w \in W_F,$$

of $\mathcal{G}'$ into ${}^LG$. In this way, we obtain a set of representatives of elliptic, θ-twisted endoscopic data for G.

A reader unfamiliar with these notions can play with the slightly more complicated examples of $G = Sp(2n)$ (for ordinary endoscopy) and $G = GL(2n)$ (for θ-twisted endoscopy). In both of these cases, $\widehat{G}'$ has a factor $SO(2m,\mathbb{C})$, while the corresponding factor of the full centralizer of s' is the full orthogonal group $O(2m,\mathbb{C})$. Unlike in the case of $O(2m+1,\mathbb{C})$ above, the nonidentity component of $O(2m,\mathbb{C})$ acts by outer automorphism on $SO(2m,\mathbb{C})$. The required choice of η' then defines the factor $SO(2m,\eta')$ of G' as a quasisplit outer twist over F of the split group $SO(2m)$, as well as the L-embedding ξ' of the group

$$\mathcal{G}' = {}^LG' = \widehat{G}' \rtimes W_F$$

into ${}^LG$. Slightly more complicated still is the example that G is a quasisplit group $SO(2n,\eta)$. In this case,

$$\widehat{G}' = SO(2m,\mathbb{C}) \times SO(2n-2m,\mathbb{C})$$

and

$$\widehat{G}' = SO(2m,\eta') \times SO(2n-2m,\eta''),$$

for characters η' and η'' with $\eta'\eta'' = \eta$.

In the general case, $\mathcal{G}'$ is always a split extension of W_F by $\widehat{G}'$, but it may not be L-isomorphic with the L-group of G'. Even if it is, it may not come with a preferred L-isomorphism. For these reasons, one has in general to equip the endoscopic datum represented by G' with an auxiliary datum $(\widetilde{G}',\widetilde{\xi}')$. The first component is a suitable central extension $\widetilde{G}'$ of G' over F. The second is an L-embedding $\widetilde{\xi}'$ of $\mathcal{G}'$ into ${}^L\widetilde{G}'$. (See [**KoSh**, §2.2].) This is the setting, namely that of an endoscopic datum G' (twisted or otherwise) with auxiliary datum $(\widetilde{G}',\widetilde{\xi}')$, for the transfer $f \to f'$ of functions.

We take the first function f to be in the Hecke module $\mathcal{H}(G_\theta)$ on the space

$$G_\theta(F) = G(F) \rtimes \theta$$

(relative to the fixed maximal compact subgroup K of $G(F)$). The image $f' = f^{\tilde{G}'}$ of f is a function on the set of strongly G-regular, stable conjugacy classes $\widetilde{\delta}'$ in $\widetilde{G}'(F)$. It is defined as a sum

$$f'(\widetilde{\delta}') = f'_\Delta(\widetilde{\delta}') = \sum_\gamma \Delta(\widetilde{\delta}',\gamma) f_G(\gamma)$$

over the set of strongly $G(F)$-regular conjugacy classes γ in $G_\theta(F)$. The linear form $f_G(\gamma)$ is the invariant orbital integral of f over the $G(F)$-orbit of γ, while $\Delta(\widetilde{\delta}',\gamma)$ is a Langlands-Shelstad-Kottwitz transfer factor for θ, G' and $(\widetilde{G}',\widetilde{\xi}')$. We note that for a given $\widetilde{\delta}'$, $\Delta(\widetilde{\delta}',\gamma)$ is supported on the finite set of classes γ of which $\widetilde{\delta}'$ is a norm [**KoSh**, p. 29] (or an image, in the language of [**LS**, p. 226]).

The Langlands-Shelstad-Kottwitz conjecture asserts that f' is the image of a function in a Hecke algebra on $\widetilde{G}'(F)$. More precisely, $f'(\widetilde{\delta}')$ should be the stable orbital integral at $\widetilde{\delta}'$ of a fixed function in the equivariant Hecke algebra $\mathcal{H}(\widetilde{G}',\widetilde{\eta}')$, where $\widetilde{\eta}'$ is a character on the kernel $\widetilde{C}'(F)$ of $\widetilde{G}'(F) \to G'(F)$ determined by the L-embedding $\widetilde{\xi}'$. In other words, f' should belong to the space

$$\mathcal{S}(\widetilde{G}',\widetilde{\eta}') = \big\{ h^{\tilde{G}'} : \ h \in \mathcal{H}(\widetilde{G}',\widetilde{\eta}') \big\},$$

where $h^{\tilde{G}'}$ is the function of $\widetilde{\delta}'$ given by the stable orbital integral of h. This is one of the properties Shelstad has established for real groups (at least when θ is trivial), but which remains open for p-adic F. We assume it in what follows.

The transfer factor $\Delta(\widetilde{\delta}',\gamma)$ on which the mapping $f \to f'$ rests is an explicit function (albeit one that is sophisticated enough to be pretty complicated). It is determined up to a nonzero multiplicative constant. It also depends on the choice of auxiliary datum $(\widetilde{G}',\widetilde{\xi}')$. However, a change in this datum amounts to a change in $\Delta(\widetilde{\delta}',\gamma)$ that multiplies it by a linear character in $\widetilde{\delta}'$. There is a similar equivariance property of $\Delta(\widetilde{\delta}',\gamma)$ under automorphisms of $(G',\widetilde{\delta}')$. From this it is not hard to show that the set of pairs $(\Delta,\widetilde{\delta}')$, where Δ includes the choice of auxiliary datum $(\widetilde{G}',\widetilde{\xi}')$, determines a principal $U(1)$-bundle over the set of isomorphism classes of pairs (G',δ'), where δ' is simply a strongly G-regular element in $G'(F)$, and isomorphism is in the sense [**KoSh**, p. 18] of endoscopic data. (See [**A5**].) This allows us to interpret the transfer $f'(\delta') = f'_{\Delta}(\widetilde{\delta}')$ of f as a section of the dual line bundle.

I should point out that in the twisted case there are a couple of minor differences here with the setting of [**KoSh**]. For example, the general case in [**KoSh**] is complicated by the existence of a 1-cocycle $\overline{z}_\sigma$ from W_F to a finite group $Z_{\mathrm{sc},\theta}$. When this cocycle is nontrivial, the corresponding transfer factor $\Delta(\widetilde{\delta}',\gamma)$ has a mild equivariance condition under stable conjugation of the first variable $\widetilde{\delta}'$ [**KoSh**, (5.4)]. It seems to me that the cocycle could be removed with an appropriate choice of the auxiliary datum $(\widetilde{G}',\widetilde{\xi}')$. I have not checked this point, but in any case, there is no harm in simply assuming here that the cocycle is trivial.

In [**KoSh**], the second variable for Δ is actually a (G,θ)-twisted conjugacy class in $G(F)$, rather than a $G(F)$-orbit in $G_\theta(F)$. However, the mapping $\gamma \to \gamma\theta^{-1}$, $\gamma \in G_\theta(F)$, is a bijection between the two kinds of classes. Our description here does therefore match that of [**KoSh**]. Observe that the variety G_θ is a G-bitorsor over F, in the sense that it comes with commuting left and right G-actions

$$x_1(x \rtimes \theta)x_2 = \big(x_1 x\,\theta(x_2)\big) \rtimes \theta, \qquad x_1, x_2, x \in G,$$

over F that are each simply transitive. Since it also has an F-rational point, $G_\theta(F)$ is a $G(F)$-bitorsor.

We shall need a slightly different perspective. Suppose that G_* is any G-bitorsor over F that has a rational point. Then for any point $\tau \in G_*(F)$, we obtain an automorphism $\overline{\tau}$ of G over F by setting

$$\overline{\tau}(x)\tau = \tau x, \qquad x \in G.$$

Suppose that G' is an endoscopic datum for G_* (relative to any $\overline{\tau}$), with auxiliary datum $(\widetilde{G}', \widetilde{\xi}')$, and that $\Delta(\widetilde{\delta}', \gamma)$ is a function of a strongly G-regular stable conjugacy class $\widetilde{\delta}'$ in $\widetilde{G}'(F)$ and a strongly G-regular $G(F)$-conjugacy class γ in $G_*(F)$. We shall say that Δ is *transfer factor* for G, G' and $(\widetilde{G}', \widetilde{\xi}')$ if there is a point $\tau \in G_*(F)$ such that the function

$$(\widetilde{\delta}', x \rtimes \overline{\tau}) \longrightarrow \Delta(\widetilde{\delta}', x\tau), \qquad x \in G(F),$$

is a transfer factor for $\overline{\tau}$, G' and $(\widetilde{G}', \widetilde{\xi}')$. The point τ is of course not unique. For example, we can replace it by a translate τz, for any z in the center $Z\big(G(F)\big)$ of $G(F)$. In other words, if Δ is a transfer factor for G_*, so is the function

$$(z\Delta)(\widetilde{\delta}', \gamma) = \Delta(\widetilde{\delta}', \gamma z).$$

Observe that $z\Delta$ equals Δ if z is of the form $z_1^{-1}\overline{\tau}(z_1)$ for some $z_1 \in Z\big(G(F)\big)$. It follows that the action $\Delta \to z\Delta$ factors through the quotient

$$Z\big(G(F)\big)_* = Z\big(G(F)\big)/\big\{z_1^{-1}\overline{\tau}(z_1) :\ z_1 \in Z\big(G(F)\big)\big\}$$

of $Z\big(G(F)\big)$. In fact, it can be shown that the set of pairs $(\Delta, \widetilde{\delta}')$, where Δ is a transfer factor for G_* (which comes with the data G' and $(\widetilde{G}', \widetilde{\xi}')$), determines a principal $U(1) \times Z_G(F)_*$-bundle over the set of isomorphism classes of pairs (G', δ'), where δ' is simply a strongly G-regular element in $G'(F)$.

We are assuming the Langlands-Shelstad-Kottwitz transfer conjecture. The correspondence $f \to f'$ is defined in geometric terms, which is to say by means of orbital integrals. However, it also has a conjectural spectral interpretation. We will be assuming this as well.

The spectral analogues of stable orbital integrals are parametrized by tempered Langlands parameters. These are L-homomorphisms

$$\phi :\ L_F \longrightarrow {}^L G$$

from the local Langlands group

$$L_F = \begin{cases} W_F, & \text{if } F \text{ is archimedean,} \\ W_F \times SU(2), & \text{if } F \text{ is } p\text{-adic,} \end{cases}$$

to ${}^L G$, whose image projects to a relatively compact subset of $\widehat{G}$. Langlands conjectured that these parameters, taken up to $\widehat{G}$-conjugacy, index a partition of the set of irreducible tempered representations of $G(F)$ into finite packets Π_ϕ. He also conjectured that the sum

$$f^G(\phi) = \sum_{\pi \in \Pi_\phi} f_G(\pi), \qquad f \in \mathcal{H}(G),$$

of the characters

$$f_G(\pi) = \operatorname{tr}\big(\pi(f)\big)$$

in the packet of a given ϕ should be *stable*, in the sense that it depends only on the set of stable orbital integrals of f.

Suppose now that G' is a twisted endoscopic datum for G (relative to an automorphism θ, or if one prefers, a G-torsor G_*). Suppose also that

$$\phi' :\ L_F \longrightarrow \mathcal{G}'$$

is an L-isomorphism with relatively compact image in $\widehat{G}'$. Then the composition $\phi = \xi' \circ \phi'$ is a tempered Langlands parameter for G. (We are assuming that all L-embeddings are of unitary type, so that the image of ϕ in $\widehat{G}$ is indeed relatively compact.) Similarly, if $(\widetilde{G}', \widetilde{\xi}')$ is an auxiliary datum for G', the composition $\widetilde{\phi}' = \widetilde{\xi}' \circ \phi'$ is a tempered Langlands parameter for $\widetilde{G}'$. Suppose that $f' = f'_\Delta$ is the image in $\mathcal{S}(\widetilde{G}', \widetilde{\eta}')$ of $f \in \mathcal{H}(G)$, relative to a given transfer factor Δ. Since the stable linear form attached to $\widetilde{\phi}'$ depends only on the image in $\mathcal{S}(\widetilde{G}', \widetilde{\eta}')$ of a given test function in $\mathcal{H}(\widetilde{G}', \widetilde{\eta}')$, our assumptions imply the existence of a well defined linear form

$$f'(\phi') = f'(\widetilde{\phi}'), \qquad f \in \mathcal{H}(G_*),$$

on $\mathcal{H}(G)$. What is it?

We are dealing here with the remaining part of Langlands' conjectural characterization of the packets Π_ϕ. In the untwisted case, the assertion is that $f'(\phi')$ is again a linear combination of characters of representations $\pi \in \Pi_\phi$, but with nontrivial coefficients $\Delta(\phi', \pi)$ that depend in the appropriate way on the choice of Δ. There is in fact a conjectural interpretation of these coefficients in terms of characters on a natural finite group attached to ϕ, but this need not concern us here.

What does concern us is the twisted analogue of this conjectural assertion. We assume now that we are working with a general bitorsor G_*. Then G_* determines a permutation $\pi \to *\pi$ on the set of (equivalence classes of) irreducible representations of $G(F)$, defined by setting

$$(*\pi)(x) = \pi\big(\overline{\tau}^{-1}(x)\big), \qquad x \in G(F),$$

on any $\tau \in G(F)$. There is a dual permutation on the set of tempered Langlands parameters that leaves ϕ fixed, since ϕ' factors through the twisted endoscopic datum G'. This means that the L-packet Π_ϕ will be preserved by the original permutation. Let Π^*_ϕ be the subset of representations $\pi \in \Pi_\phi$ such that $*\pi = \pi$. Then π belongs to this subset if and only if it has an extension π_* to $G_*(F)$, which is to say, a mapping π_* of $G_*(F)$ into the set of unitary operators on the space on which π acts such that

$$\pi_*(x_1 \tau x_2) = \pi(x_1)\pi_*(\tau)\pi(x_2), \qquad \tau \in G_*(F),\ x_1, x_2 \in G(F).$$

(We are using the fact here that a tempered representation π is unitary.) The extension π_* is of course not unique. The fibres of the mapping $\pi_* \to \pi$ are $U(1)$-torsors, under the obvious action

$$(u\pi_*)(\tau) = u\pi_*(\tau), \qquad u \in U(1),\ \tau \in G_*(F)$$

of the group $U(1)$.

We can now describe the general conjectural expansion of the linear form above. It is

$$f'(\phi') = \sum_{\pi \in \Pi^*_\phi} \Delta(\phi', \pi_*) f_G(\pi_*), \qquad f \in \mathcal{H}(G), \tag{3.1}$$

for coefficients $\Delta(\phi', \pi_*)$ that vary in the appropriate way with respect to the original transfer factor Δ, and that satisfy

$$\Delta(\phi', u\pi_*) = \Delta(\phi, \pi_*)u^{-1}, \qquad u \in U(1).$$

Observe that the summand in (3.1) depends only on π, and not the extension π_*.

As the notation suggests, $f'(\phi')$ depends only on ϕ'. In other words, it is independent of the choice of auxiliary datum $(\widetilde{G}', \widetilde{\xi}')$. A variation in the choice of $\widetilde{\xi}'$, for example, is compensated by a corresponding variation in the composition $\widetilde{\phi}' = \widetilde{\xi}' \circ \phi'$. However, $f'(\phi') = f'_\Delta(\phi')$ does still depend on the choice of transfer factor Δ. We can think of Δ as a family of transfer factors, one for each choice of $(\widetilde{G}', \widetilde{\xi}')$, which satisfy natural compatibility conditions as these choices vary. Moreover, we have

$$f'_{z\Delta}(\phi') = \chi_{\phi'}(z) f'_\Delta(\phi'), \qquad z \in Z\big(G(F)\big)_*,$$

for a character $\chi_{\phi'}$ on $Z\big(G(F)\big)_*$ attached to ϕ'. One way to say this is that the set of pairs (Δ, ϕ') determines a principal $U(1)$-bundle over the set of isomorphism classes of pairs (G', ϕ').

4. Refinement of the problem

We return to the question at the end of §2. We shall state a conjectural formula that relates the intertwining trace expression (2.4) with the transfer of functions (3.1).

The expression (2.4) comes with a Levi subgroup $M \subset G$ of G and an irreducible representation π of $M(F)$. We are treating only the simplest of cases, in which π belongs to the relative discrete series $\Pi_2(M)$ of M. In particular, π lies in the L-packet Π_{ϕ_M} of a tempered Langlands parameter ϕ_M for M.

We fix a Levi subgroup ${}^LM \subset {}^LG$ of ${}^LG$ that is dual to M. In other words, we fix an L-isomorphism of the L-group of M with a Levi subgroup of ${}^LG$. This also provides us with an isomorphism of the Weyl group $W(M)$ with $W({}^LM)$, the subgroup of elements in the Weyl group $W(\widehat{M})$ that commute with the L-action of Γ_F on $\widehat{M}$. Let ϕ be the composition of ϕ_M with the L-embedding of ${}^LM$ into ${}^LG$. Then ϕ is a tempered Langlands parameter for G.

We write

$$\overline{S}_\phi = S_\phi / Z(\widehat{G})^{\Gamma_F},$$

where S_ϕ is the centralizer in $\widehat{G}$ of the image of (a representative of) ϕ, and $Z(\widehat{G})^{\Gamma_F}$ is the subgroup of elements in the center of $\widehat{G}$ that are invariant under the Galois group Γ_F. Then $\overline{S}_\phi$ is a complex (not necessarily connected) reductive group, which plays an important role in the spectral interpretation of endoscopic transfer. Let $\overline{T}_\phi$ be a fixed maximal torus in the connected component $\overline{S}^0_\phi$ of 1 in $\overline{S}_\phi$. There is then a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & W_\phi^0 & = & W_\phi^0 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{S}_\phi^1 & \longrightarrow & \mathfrak{N}_\phi & \longrightarrow & W_\phi & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{S}_\phi^1 & \longrightarrow & \mathcal{S}_\phi & \longrightarrow & R_\phi & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

of finite groups, defined as in [**A2**, §7]. Thus,

$$\mathfrak{N}_\phi = \overline{N}_\phi / \overline{T}_\phi = \pi_0(\overline{N}_\phi),$$

where $\overline{N}_\phi$ is the normalizer of $\overline{T}_\phi$ in $\overline{S}_\phi$, while $\mathcal{S}_\phi^1$ is the subgroup of components in $\overline{N}_\phi$ that commute with $\overline{T}_\phi$, and $W_\phi = \mathfrak{N}_\phi / \mathcal{S}_\phi^1$ in the Weyl group of $(\overline{S}_\phi, \overline{T}_\phi)$. There is also a surjective mapping from $\mathfrak{N}_\phi$ onto the group of connected components

$$\mathcal{S}_\phi = \overline{S}_\phi / \overline{S}_\phi^0 = \pi_0(\overline{S}_\phi)$$

of $\overline{S}_\phi$, whose kernel is the Weyl group W_ϕ^0 of $(\overline{S}_\phi^0, \overline{T}_\phi)$. The quotient

$$R_\phi = \mathcal{S}_\phi / \mathcal{S}_\phi^1 = W_\phi / W_\phi^0$$

is called the R-group of ϕ. It governs the reducibility of induced representations

$$\mathcal{I}_P(\pi), \qquad \pi \in \Pi_{\phi_M}.$$

The expression (2.4) also comes with a Weyl element $w \in W(M)$. This element represents a coset of M in G, which we regard as an M-bitorsor M_w. We can assume that $w\pi \cong \pi$, since the trace in (2.4) would otherwise vanish. It follows from the fact that π belongs to Π_{ϕ_M} that the dual element $\widehat{w}$ of w in $W(\widehat{M})$ stabilizes the Langlands parameter ϕ_M. In other words, $\widehat{w}$ belongs to the group W_ϕ in the diagram. Its preimage in $\mathfrak{N}_\phi$ is then a coset $\mathcal{S}_\phi^1 \widehat{w}$ of $\mathcal{S}_\phi^1$ in $\mathfrak{N}_\phi$. It is not hard to show that $\overline{T}_\phi$ is isomorphic to $Z(\widehat{M})^{\Gamma_F} / Z(\widehat{G})^{\Gamma_F}$, a fact that allows us to identify $\mathcal{S}_\phi^1$ with the group

$$\mathcal{S}_{\phi_M} = \pi_0(\overline{S}_{\phi_M}) = \pi_0\big(S_{\phi_M} / Z(\widehat{M})^{\Gamma_F}\big)$$

attached to ϕ_M.

Suppose that s'_w belongs to $\mathcal{S}_{\phi_M} \widehat{w}$. Then s'_w represents a semisimple coset of $Z(\widehat{M})^{\Gamma_F}$ in $\widehat{G}$. We write $\widehat{M}'$ for its connected centralizer in $\widehat{M}$. We then let ξ'_w be the identity L-embedding of the L-subgroup

$$\mathfrak{M}' = \widehat{M}' \, \phi_M(L_F)$$

into ${}^L M$. Finally, we take M' to be a quasisplit group over F that is in duality with $\widehat{M}'$ (with respect to the L-action determined by $\mathfrak{M}'$). Then M' represents a twisted endoscopic datum $(M', \mathfrak{M}', s'_w, \xi'_w)$ for M, relative to the bitorsor M_w.

We write ϕ'_w for the L-homomorphism from L_F to $\mathfrak{M}'$ whose composition with ξ'_w is ϕ_M. According to the discussion at the end of §3, we can attach a finite linear combination of traces

$$\sum_{\pi \in \Pi^w_{\phi_M}} \Delta_w(\phi'_w, \pi_w)\mathrm{tr}\big(R_P(\pi_w), \mathcal{I}_P(\pi, f)\big)$$

to any transfer factor Δ_w for M_w and M'. As in (3.1), the summand of π is independent of the choice of extension π_w of π to $M_w(F)$. It is a part of the conjectural spectral interpretation of endoscopic transfer (the part we did not state, having to do with the group $\mathcal{S}_{\phi_M}$) that by varying s'_w, we can invert the matrix of coefficients $\{\Delta_w(\phi'_w, \pi_w)\}$. The original problem is then equivalent to finding a formula for this linear combination.

Given M', we shall introduce a family of (ordinary) endoscopic data $\mathcal{E}_{M'}(G)$ for G. Suppose that s' is an element in the subset

$$s'_w \overline{T}_\phi = s'_w Z(\widehat{M})^{\Gamma_F}/Z(\widehat{G})^{\Gamma_F}$$

of $\widehat{G}/Z(\widehat{G})^{\Gamma_F}$. Copying the construction above, we take $\widehat{G}'$ to be the connected centralizer of s' in $\widehat{G}$, ξ' to be the identity embedding of the L-subgroup

$$\mathcal{G}' = \widehat{G}'\mathfrak{M}' = \widehat{G}'\phi(L_F)$$

into ${}^LG$, and G' to be a quasisplit group over F that is duality with $\widehat{G}'$ (with respect to the L-action defined by $\mathcal{G}'$). Then G' represents an endoscopic datum $(G', \mathcal{G}', s', \xi')$ for G. We define $\mathcal{E}_{M'}(G)$ to be the set of endoscopic data G' for G obtained in this way, as s' ranges over $\overline{T}_\psi s'_w$. For any G' in $\mathcal{E}_{M'}(G)$, we write ϕ' for the L-homomorphism from L_F to $\mathcal{G}'$ whose composition with ξ' equals ϕ.

Refined Problem. *Suppose that M, ϕ_M, w, and M' are fixed as above, and that G' belongs to $\mathcal{E}_{M'}(G)$. Show that there is a natural mapping*

$$\Delta \longrightarrow \Delta_w,$$

from transfer factors Δ for G and G' to transfer factors Δ_w for M_w and M', and an explicit constant

$$c(\phi_{M,w}),$$

such that

$$f'_\Delta(\phi') = c(\phi_M, w) \sum_{\pi \in \Pi^w_{\phi_M}} \Delta_w(\phi'_w, \pi_w)\mathrm{tr}\big(R_P(\pi_w)\mathcal{I}_P(\pi, f)\big), \tag{4.1}$$

for any $f \in \mathcal{H}(G)$.

The existence of a twisted transfer factor Δ_w would put a number of difficulties to rest. It would free the earlier conjecture stated in [**A2**, §7] from its problematical dependence on Whittaker models. It would also leave each side of the putative formula (4.1) in perfect balance, relative to the extension π_w of π and to the resulting dependence of each side on the choice of Δ. The constant $c(\phi_M, w)$ should of course be independent of Δ. It ought to be easy enough to write down, given what is expected for Whittaker models [**A2**, §7], but I have not done so. Whatever its form, it will have to depend on an additive character ψ_F of F, in order to cancel the corresponding dependence of the Langlands normalizing factor in $R_P(\pi_w)$. Finally, a global product of local identities (4.1) should have the desired global form [**A3**, (5.4)], since the product of local transfer factors leads to a canonical pairing, and the product of the constants $c(\phi_M, w)$ should be 1.

At the moment, I do not have any idea how to construct a mapping $\Delta \to \Delta_w$. Whatever form it might take, it would have to reduce to that of the special case that $w = 1$. In this case, M' represents an ordinary elliptic endoscopic datum for M. The associated set $\mathcal{E}_{M'}(G)$ of endoscopic data for G is familiar for its role in the stabilization of parabolic terms in the trace formula. (See [**A4**, §3], for example.) We can identify M' with a Levi subgroup of any given $G' \in \mathcal{E}_{M'}(G)$, and then $\Delta \to \Delta_w = \Delta_M$ is just the usual restriction mapping of Langlands-Shelstad transfer factors to Levi subgroups. In general, we would want the mapping $\Delta \to \Delta_w$ to have a reasonable geometric formulation. If so, it will presumably also be some kind of restriction operation. Perhaps the second variable of Δ should range over points

$$umwn, \qquad u \in N_P(F),\ m \in M(F),\ n \in w^{-1}N_P(F)w \cap N_P(F)\backslash N_P(F),$$

modeled on the original intertwining integral, with u and n being in general position and n also being close to 1. The paper [**K**] is suggestive, but I am afraid I have not given the question sufficient thought.

In the case of quasisplit classical groups, one can work with Whittaker models for general linear groups. The arguments are global. They rely on a generalization of the fundamental lemma and the stabilization of the twisted trace formula for $GL(N)$, both of which have yet to be established. These granted, one can establish an analogue of (4.1) for quasisplit orthogonal and symplectic groups. The resulting formula does not depend explicitly on transfer factors, which can be normalized according to the convention of [**KoSh**, (5.3)]. The formula is part of a long global argument, in both proof and application, despite its local nature. In particular, the analogue of (4.1) obtained for quasisplit orthogonal and symplectic groups is an essential part of the global classification of automorphic representations of these groups.

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Department of Mathematics, University of Toronto, Toronto, Ontario M5S 2E4
E-mail address: arthur@math.toronto.edu